Spatial Data Analysis

Spatial Weights

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Outline

- Weights General Concepts
- Contiguity Weights
- Distance Based Weights
- General Weights
- Guidance

Weights General Concepts

Why Spatial Weights?

- Identification Problem
  - total number of interactions is N(N-1)/2
  - all possible unique pairs i-j
  - only N observations in a cross-section
- Incidental Parameter Problem
  - number of parameters increases in size sample, $O(N^2)$
  - more data not the solution

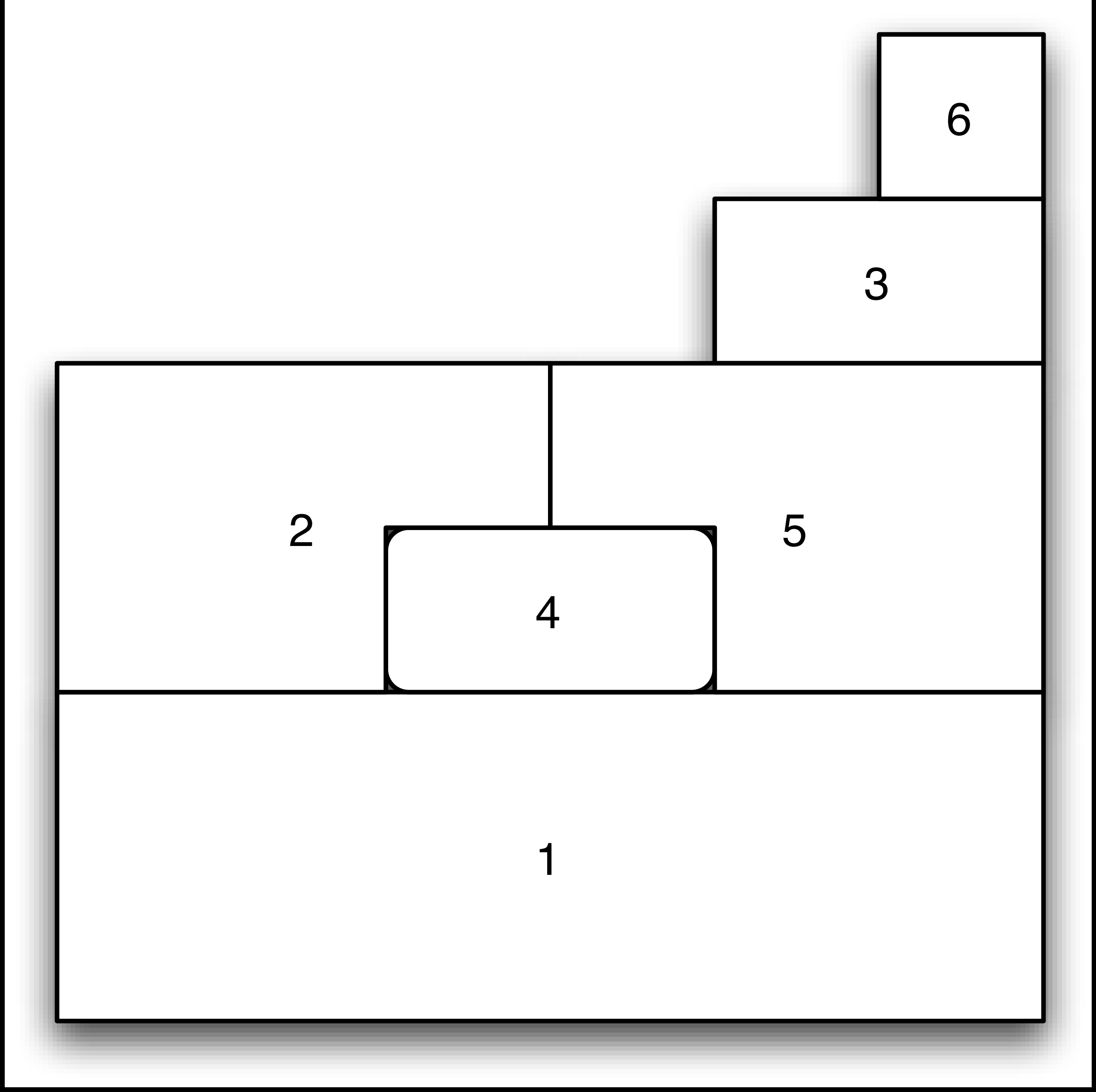
Problem

- Spatial Correlation
  - $C[y_i, y_j] \neq 0 \;\forall\; i \neq j$
  - $N(N-1)/2$ covariances
  - Only N observations

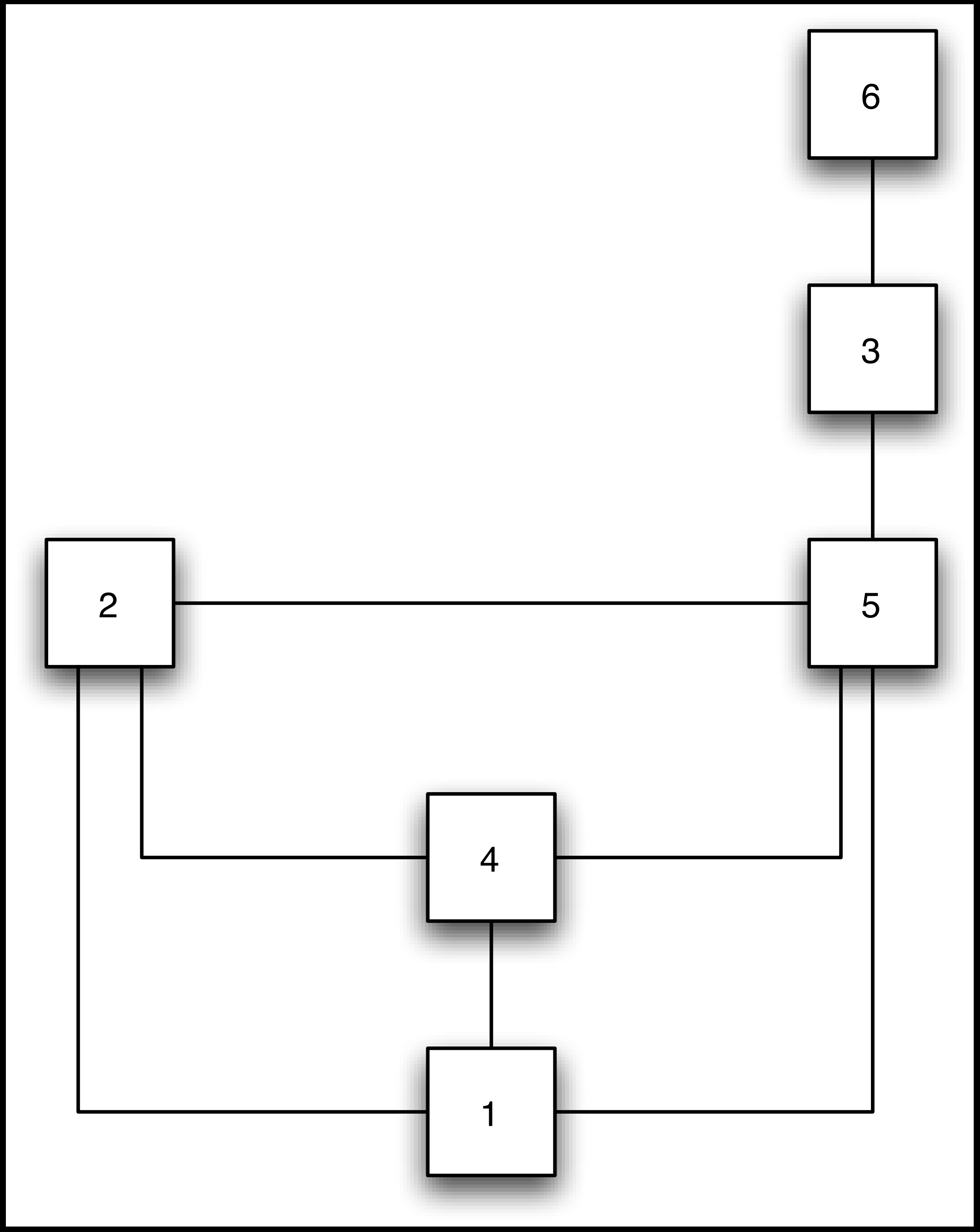
Solution

- Impose Structure of the Problem
  - set some interactions to zero
  - only “neighbors” interact directly
  - constrain the number of neighbors
- Assume a Single Parameter
  - spatial autocorrelation coefficient

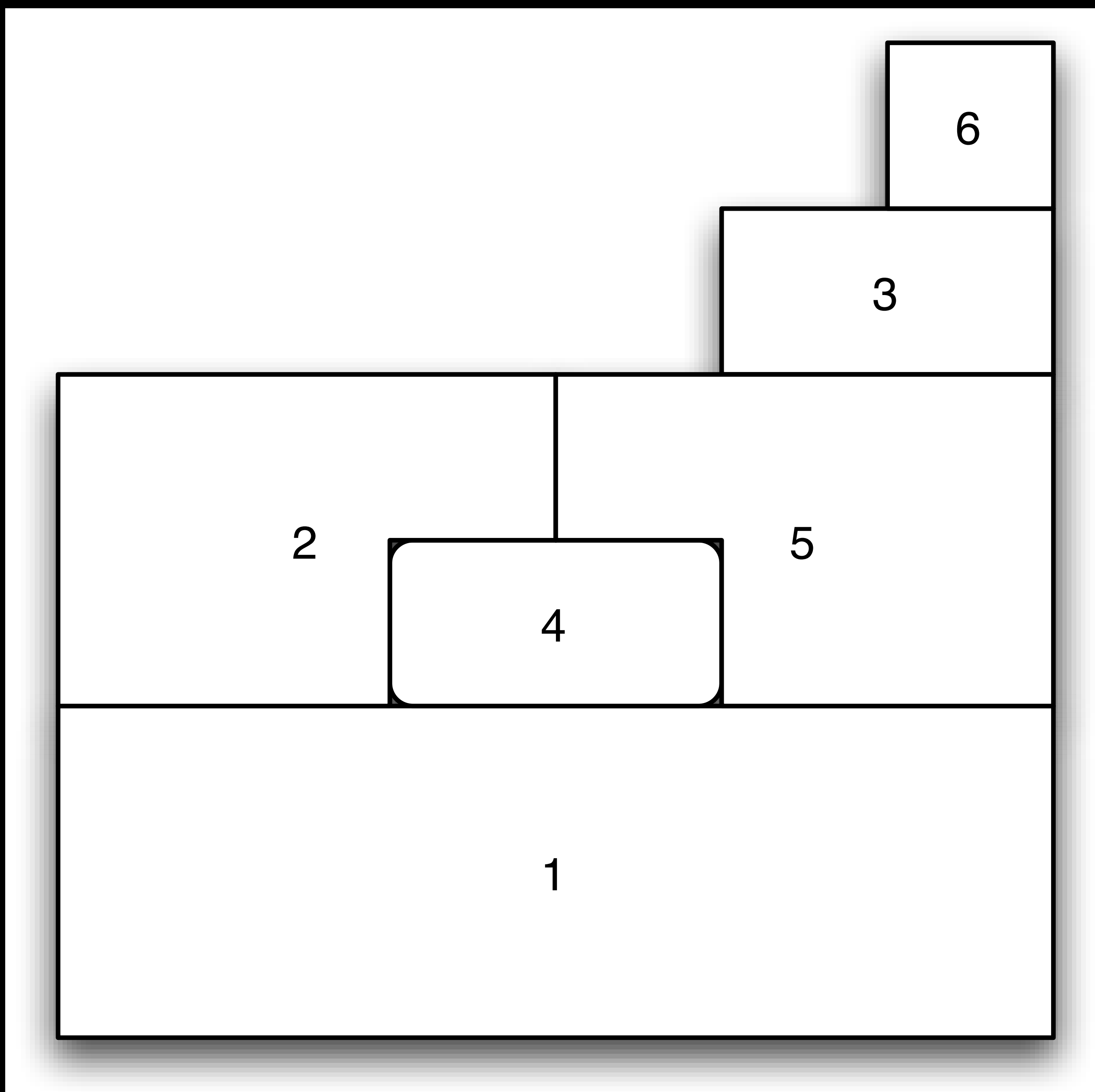
Irregular Lattice (Polygons)



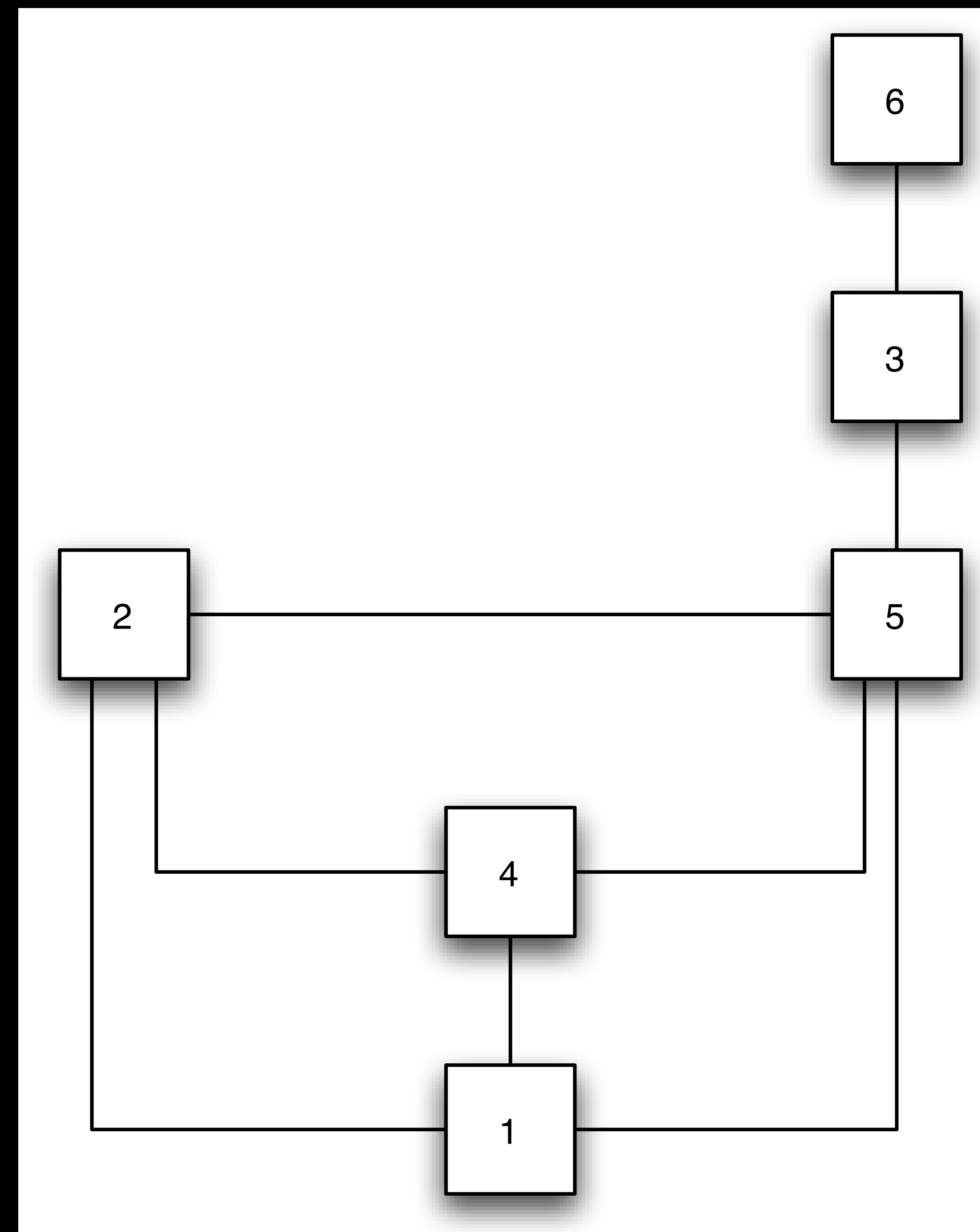
Neighbor Structure as a Graph



Map



Graph



Spatial Weights Matrix

- Definition
  - N by N positive matrix W, elements w_{ij}
  - w_{ij} nonzero for neighbors, 0 otherwise
  - $w_{ii} = 0$, no self-neighbors

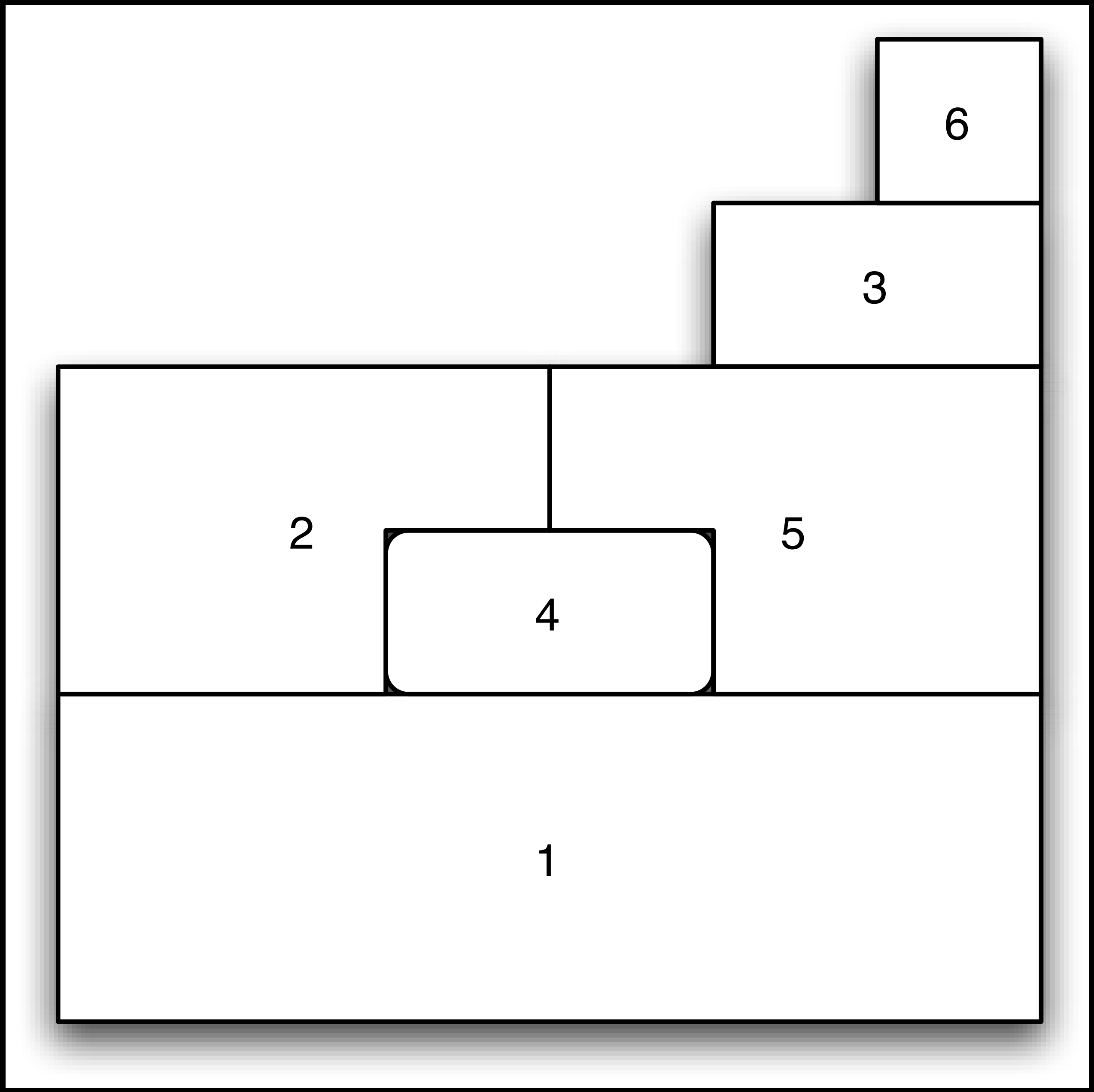
$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \dots & w_{nn} \end{bmatrix}$$

Binary Contiguity

- Weights are 0 or 1
  - for i,j neighbors, w_{ij} = 1
- Example, using common boundary

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Example



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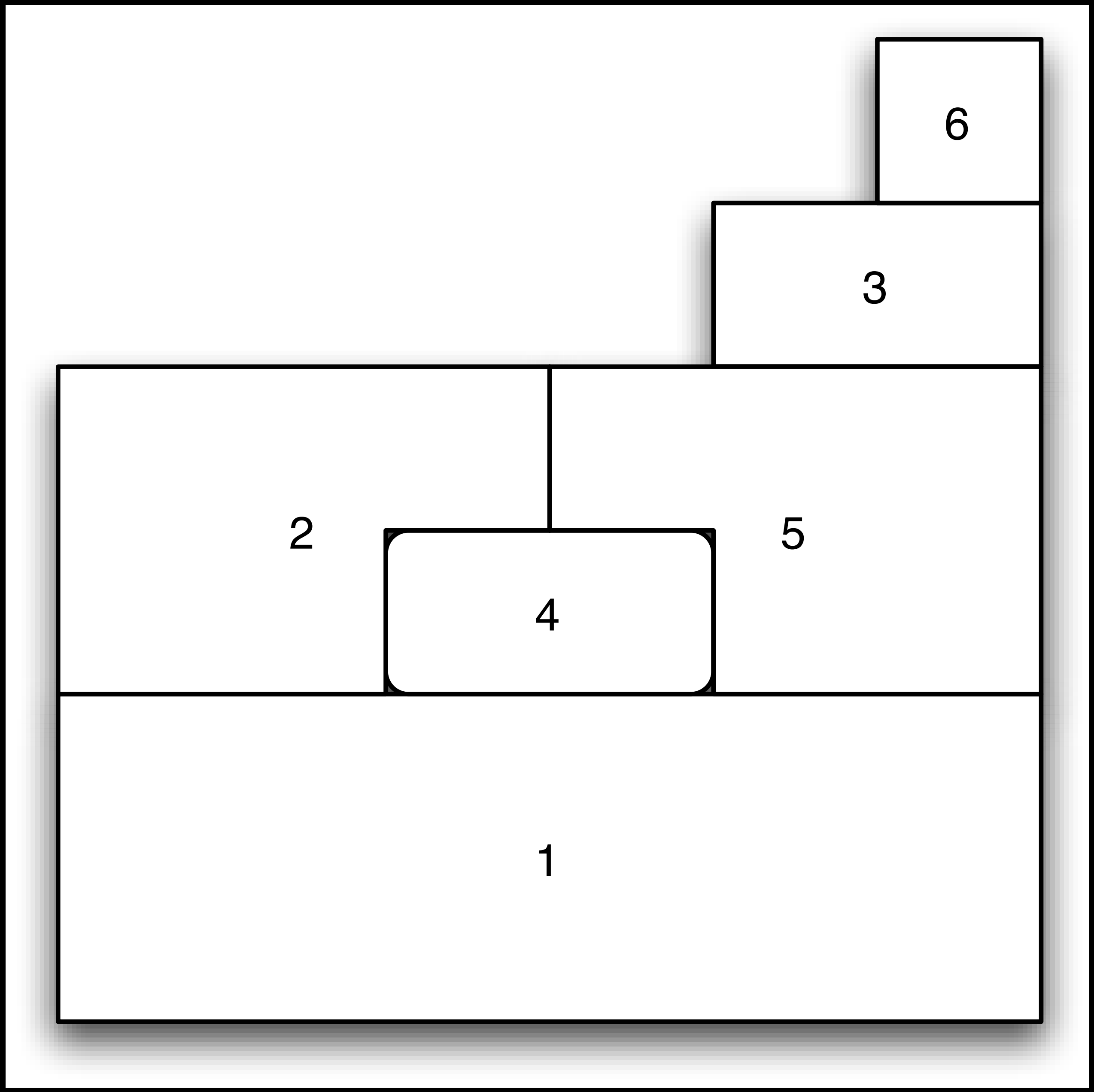

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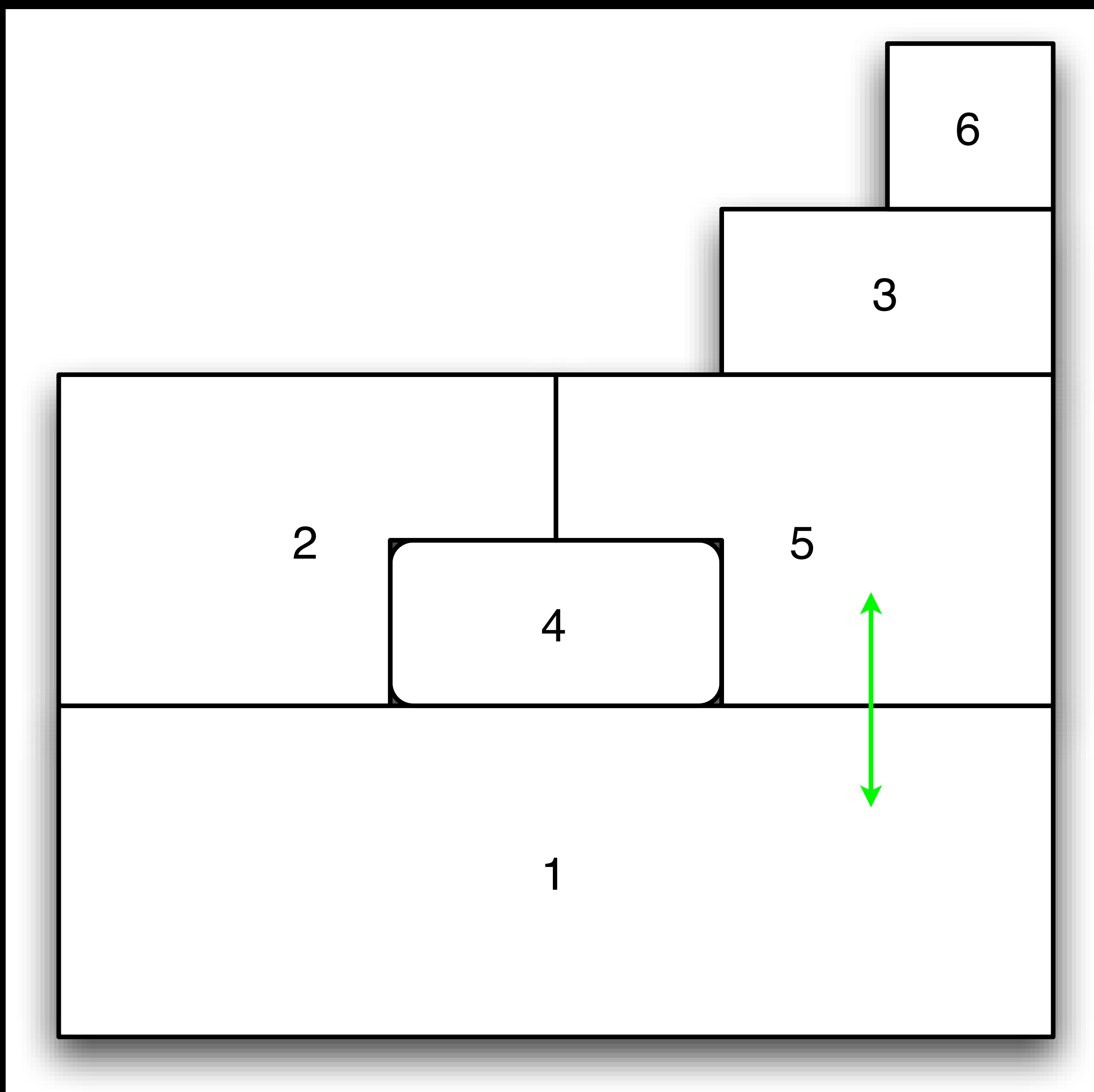

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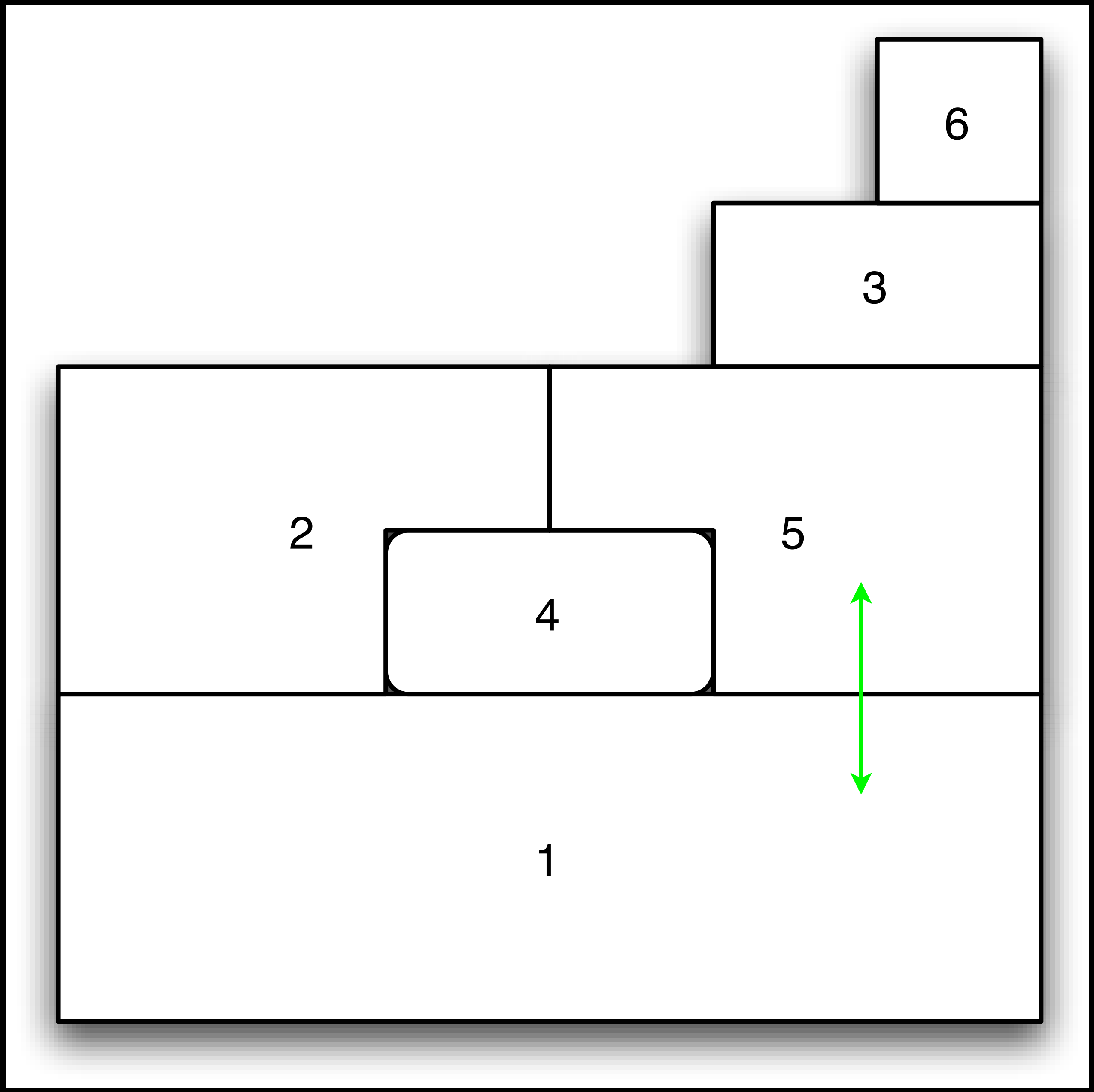

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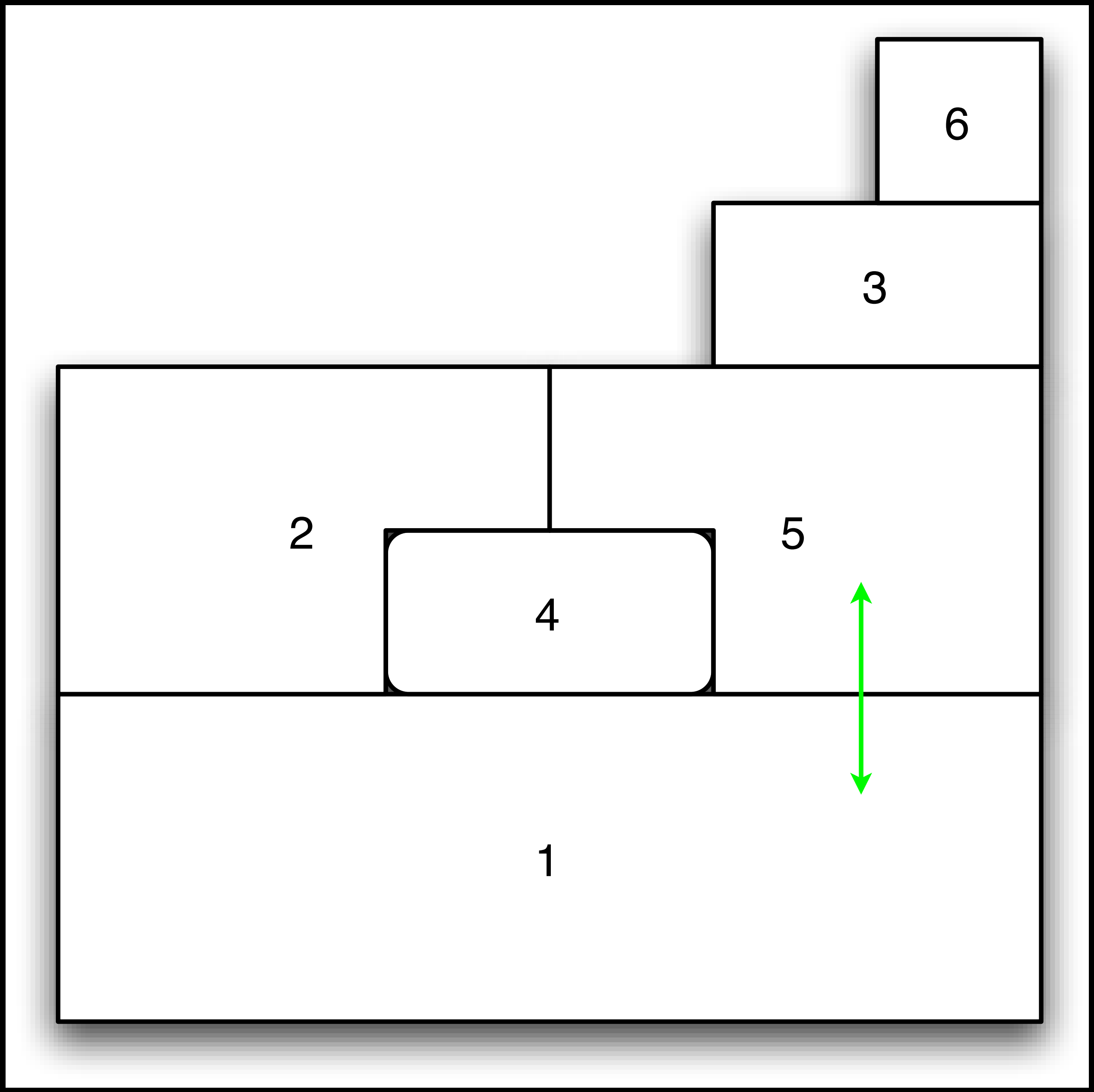

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How to Define W

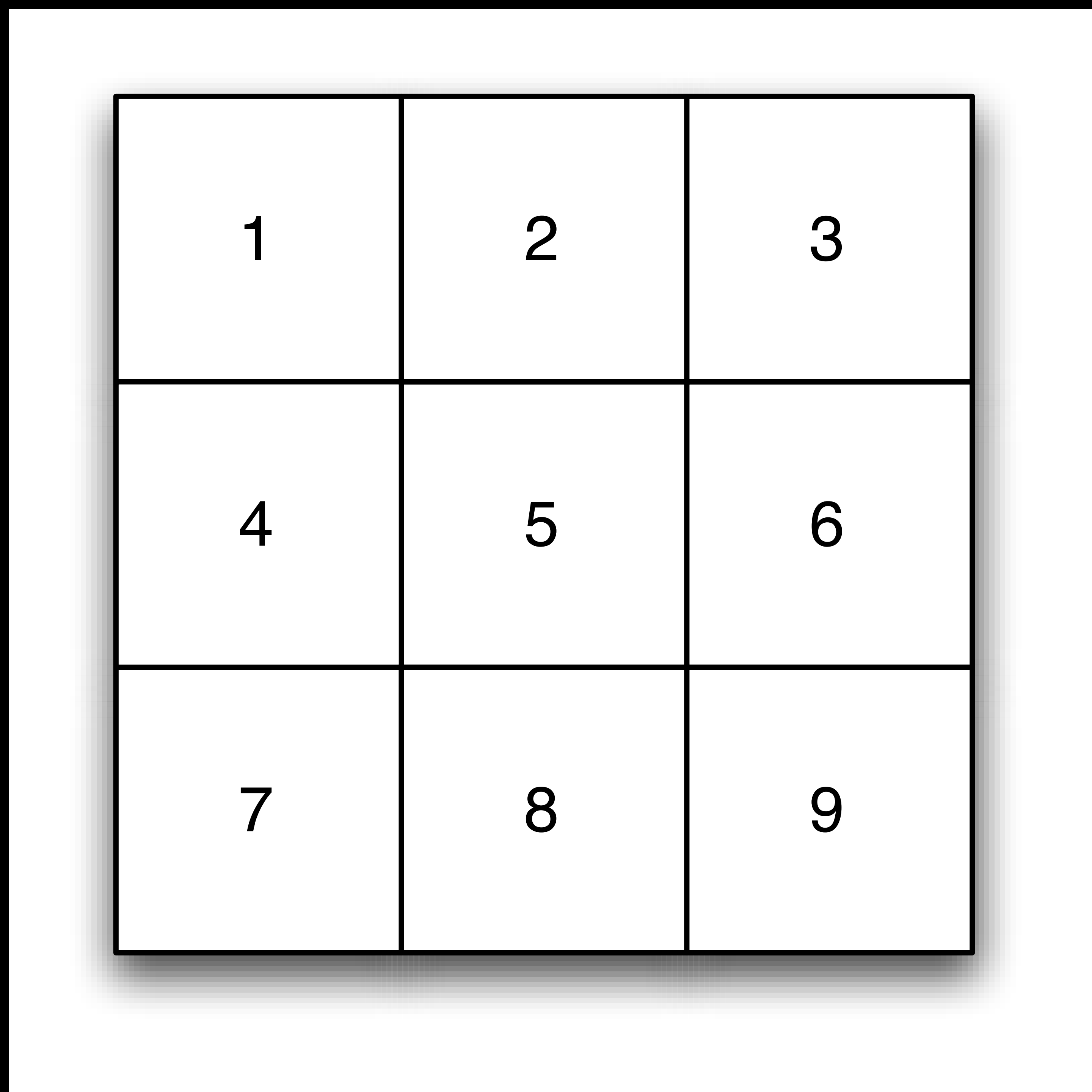
- Geographic Weights
  - contiguity
  - distance
  - general
  - graph-based weights
- Socio-Economic Weights

Contiguity

Contiguity Weights

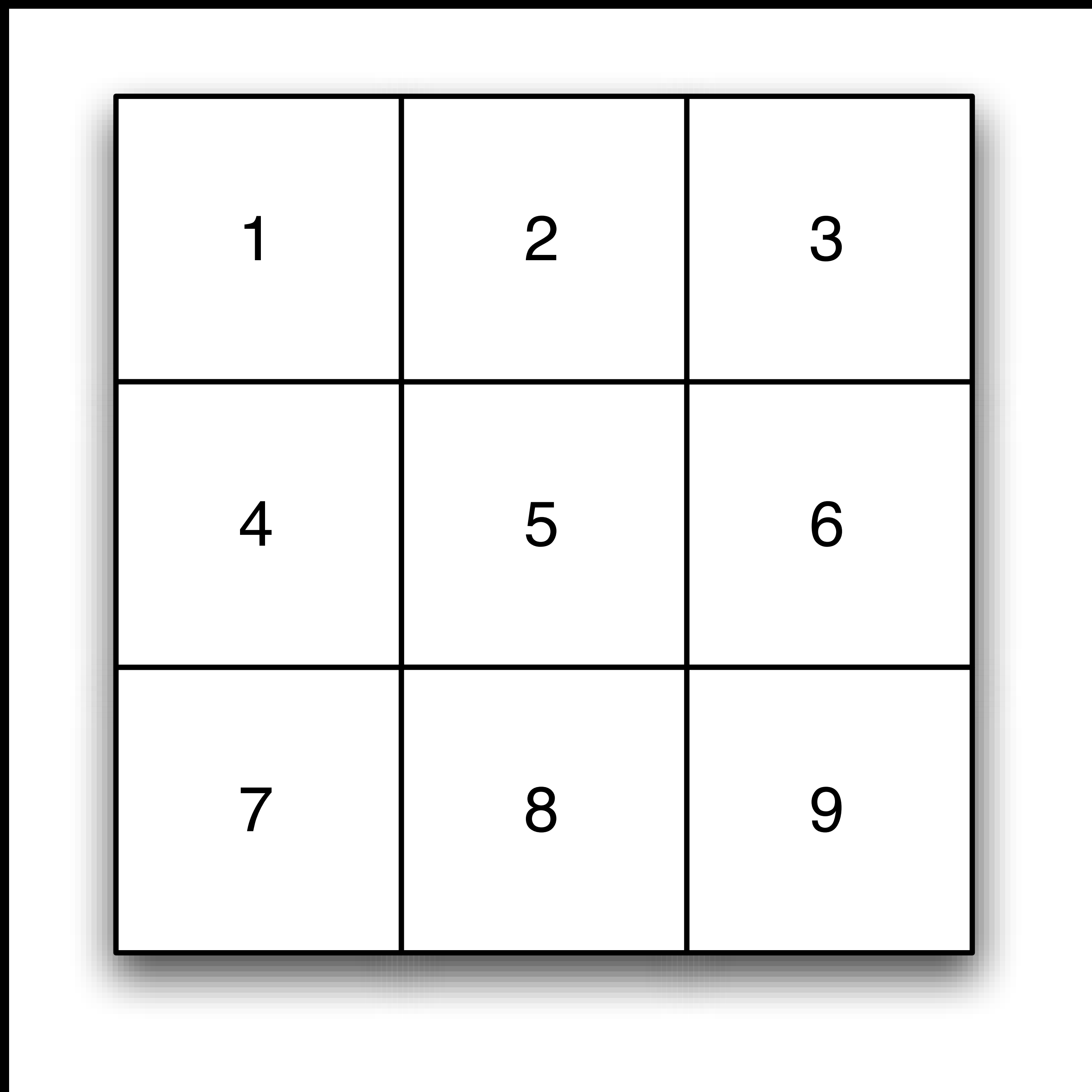
- Contiguity
    - sharing a common boundary of non-zero length
- What is a Non-zero Boundary?
- Three Views of Contiguity
    - rook
    - bishop
    - queen

Example Layout: Regular Lattice N=9



Example Layout: Regular Lattice N=9

Focus on Central Location



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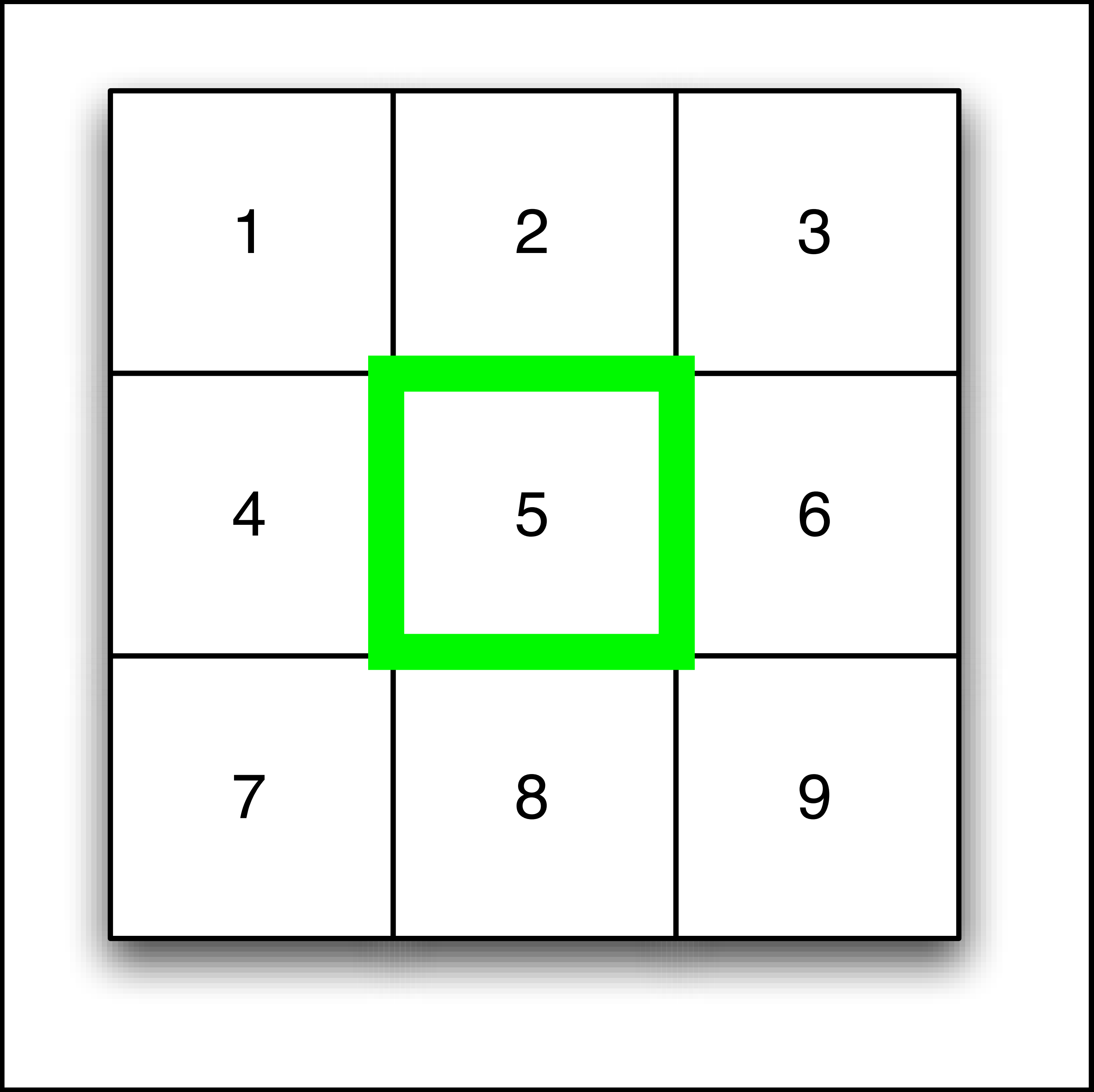
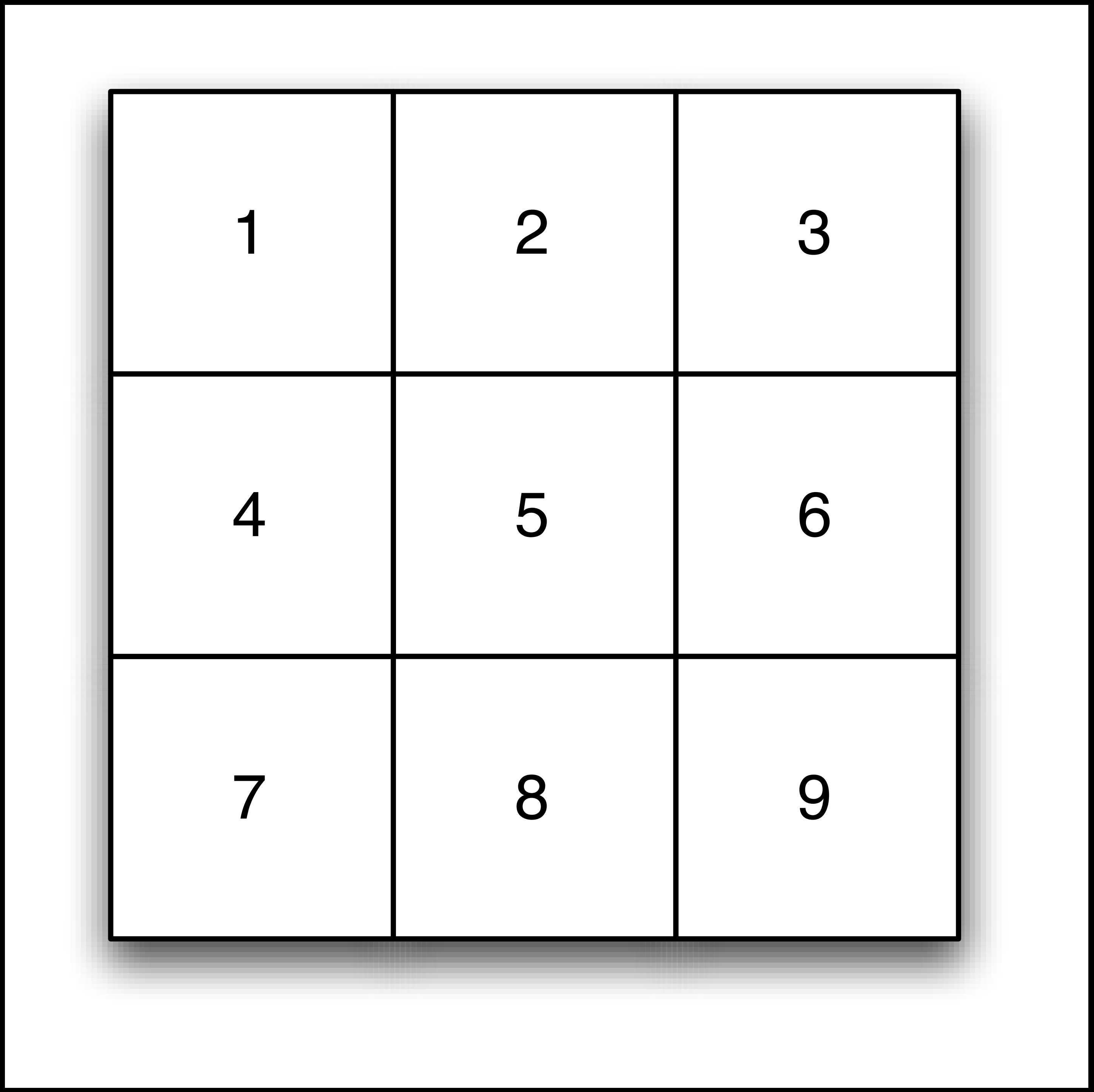


Rook Contiguity



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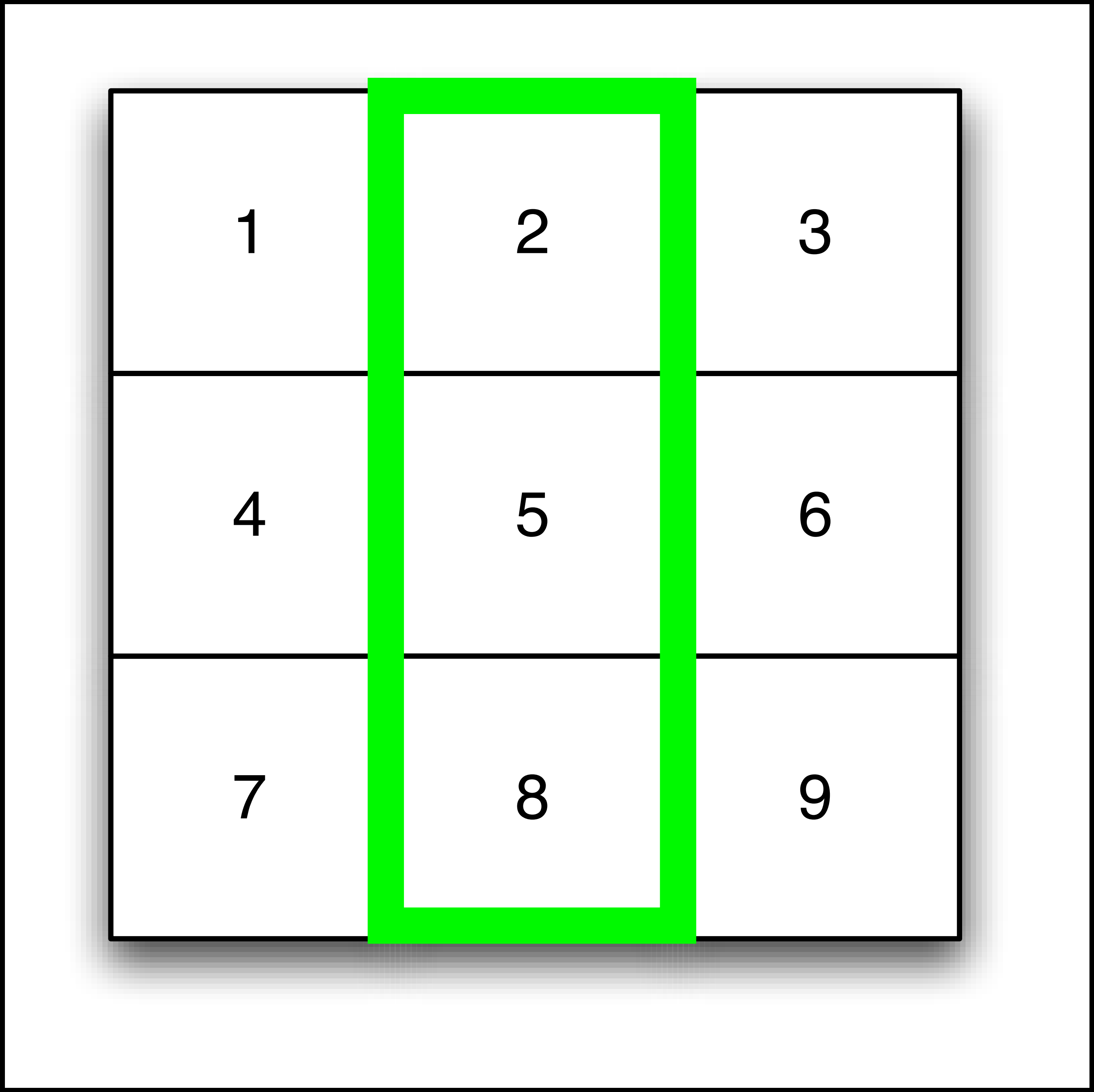

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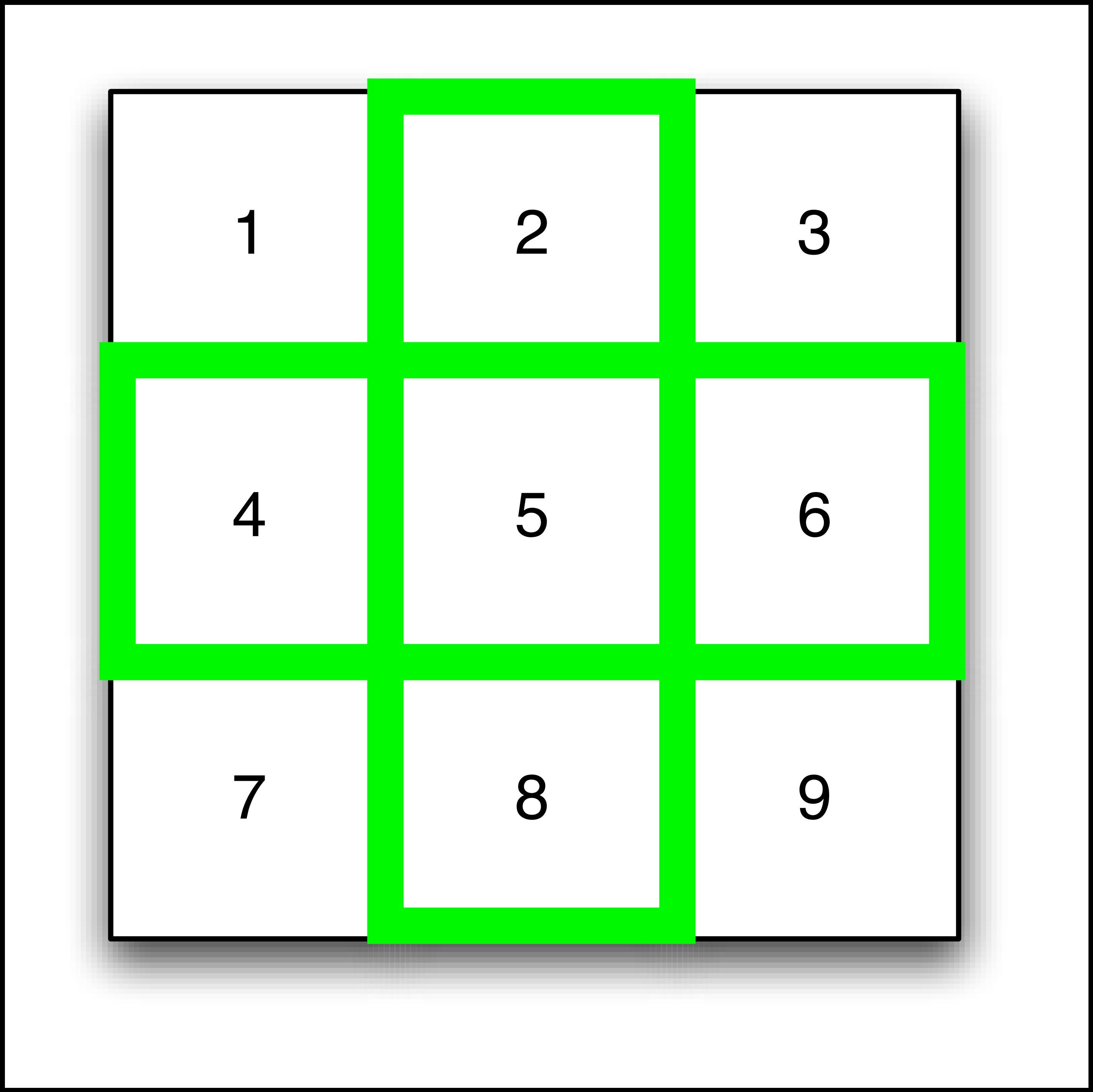

Rook Weights

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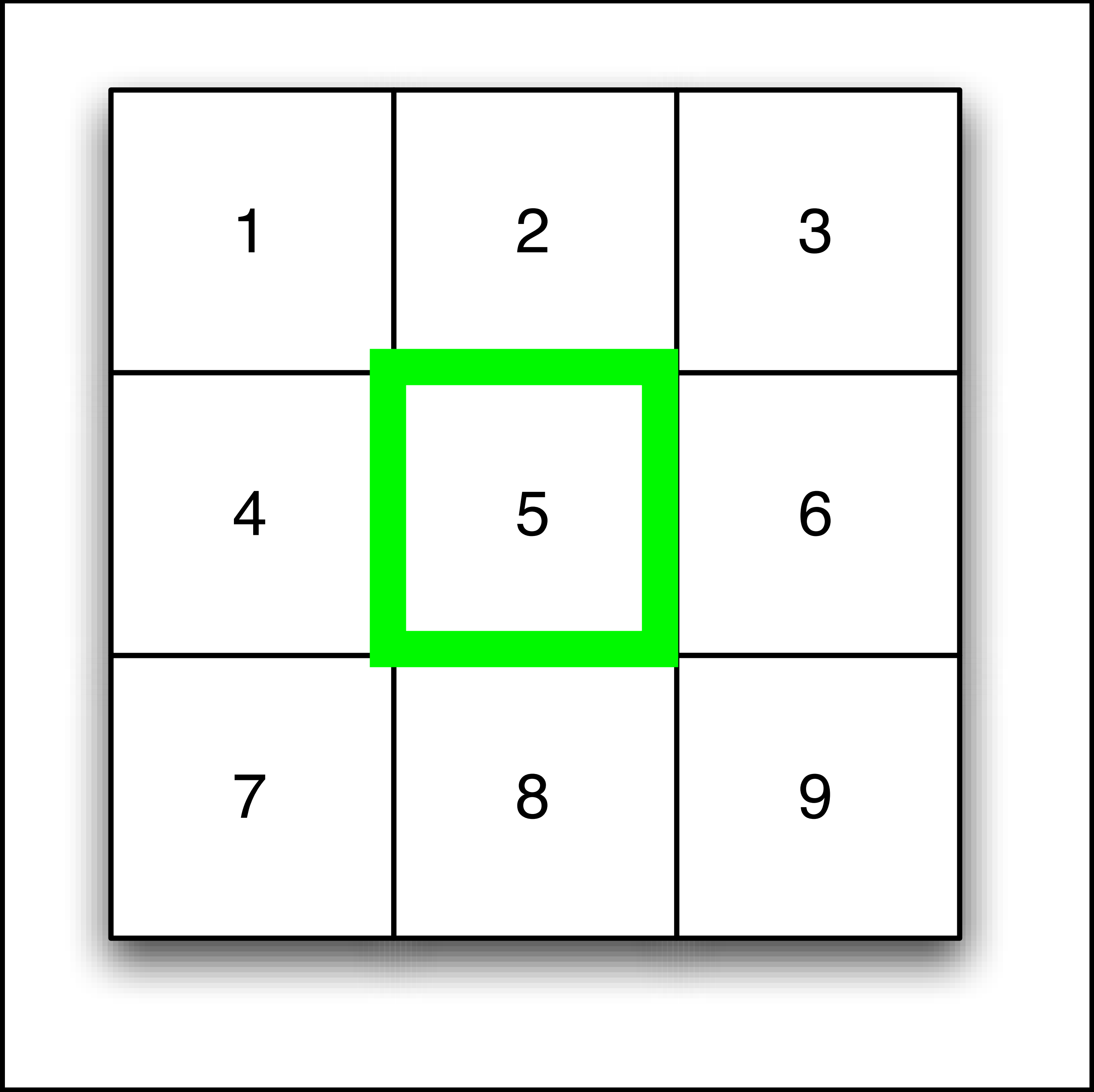
Neighbors for 5: 2, 4, 6, 8

Bishop Contiguity



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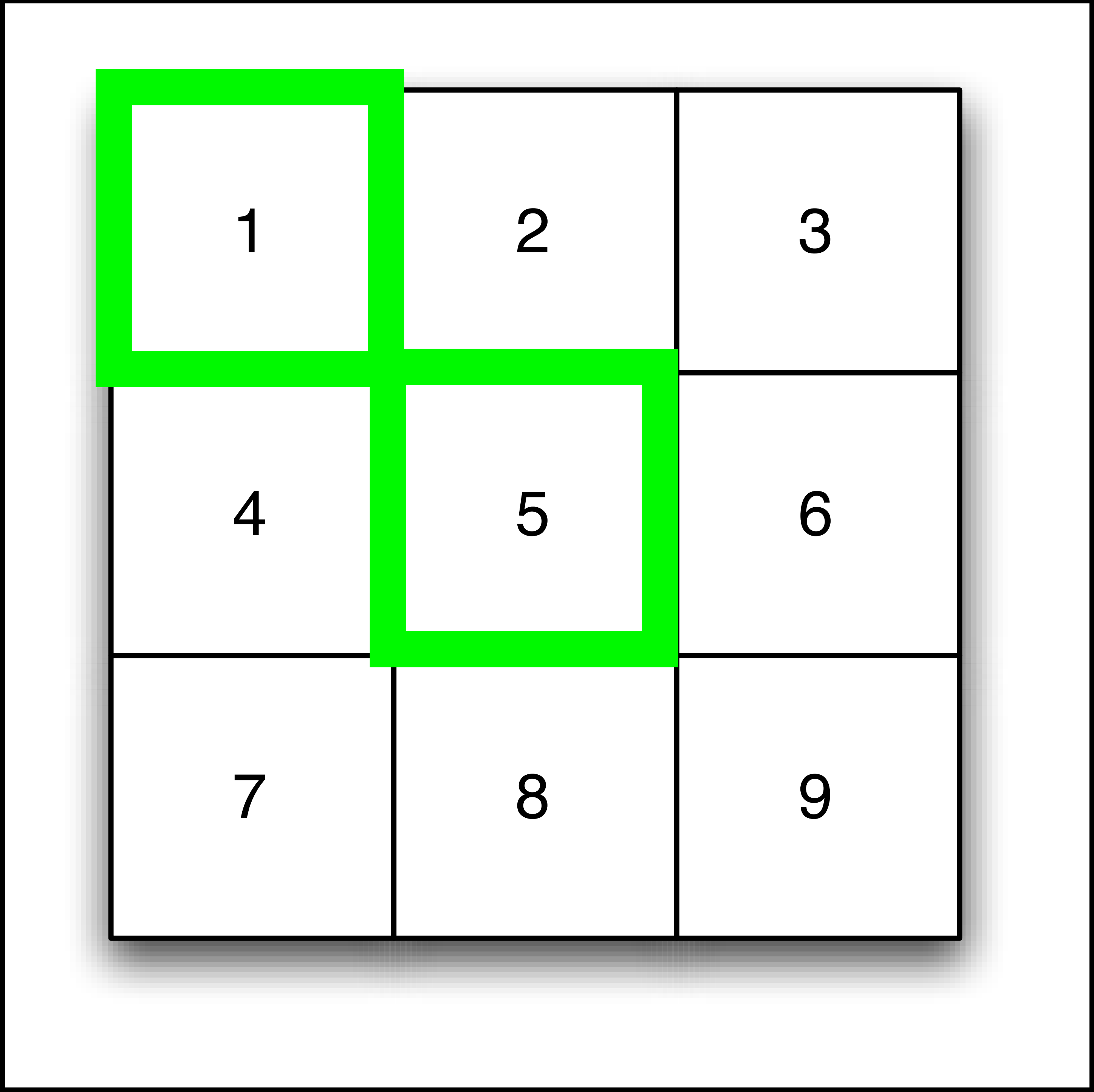

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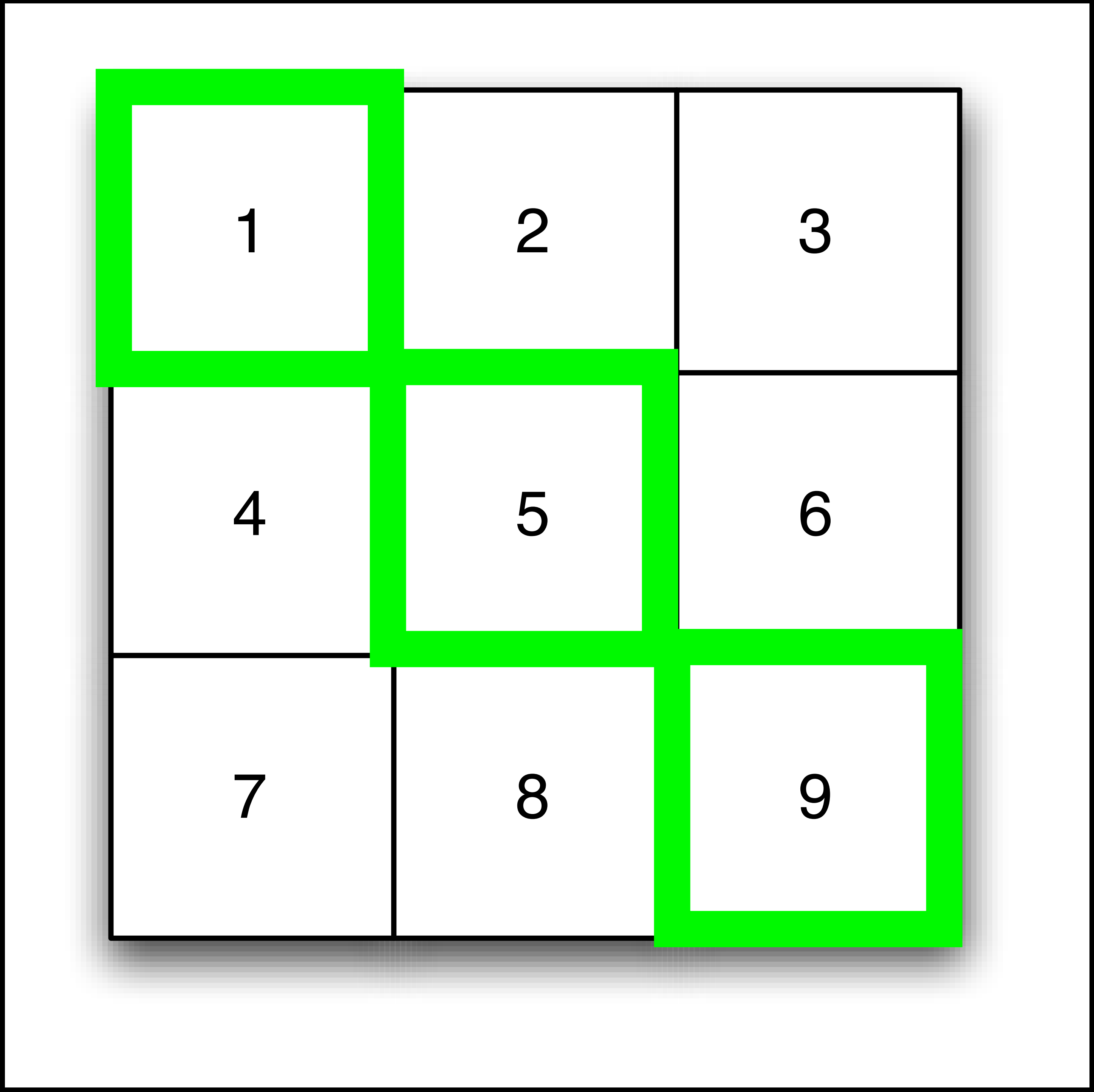

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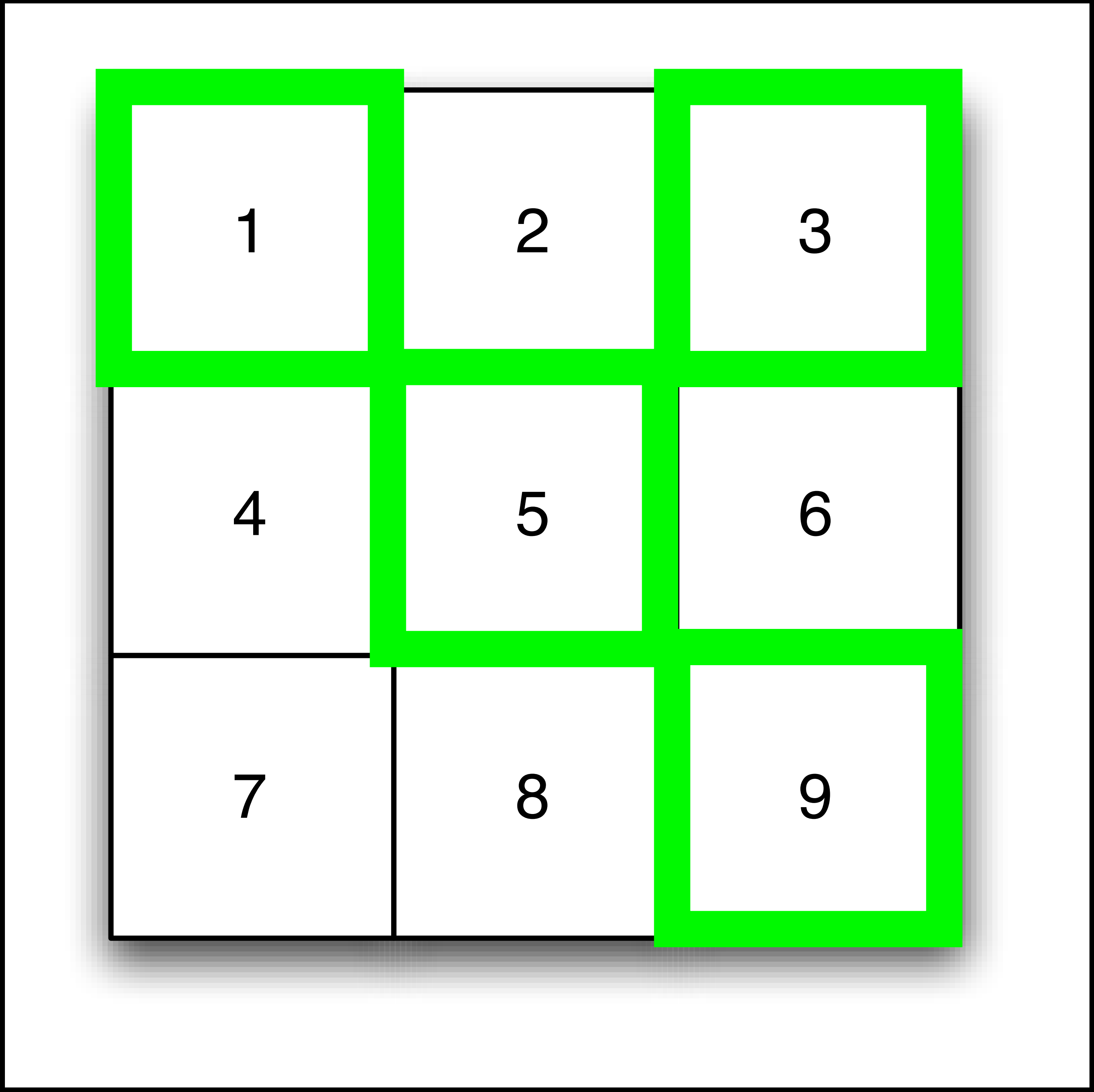

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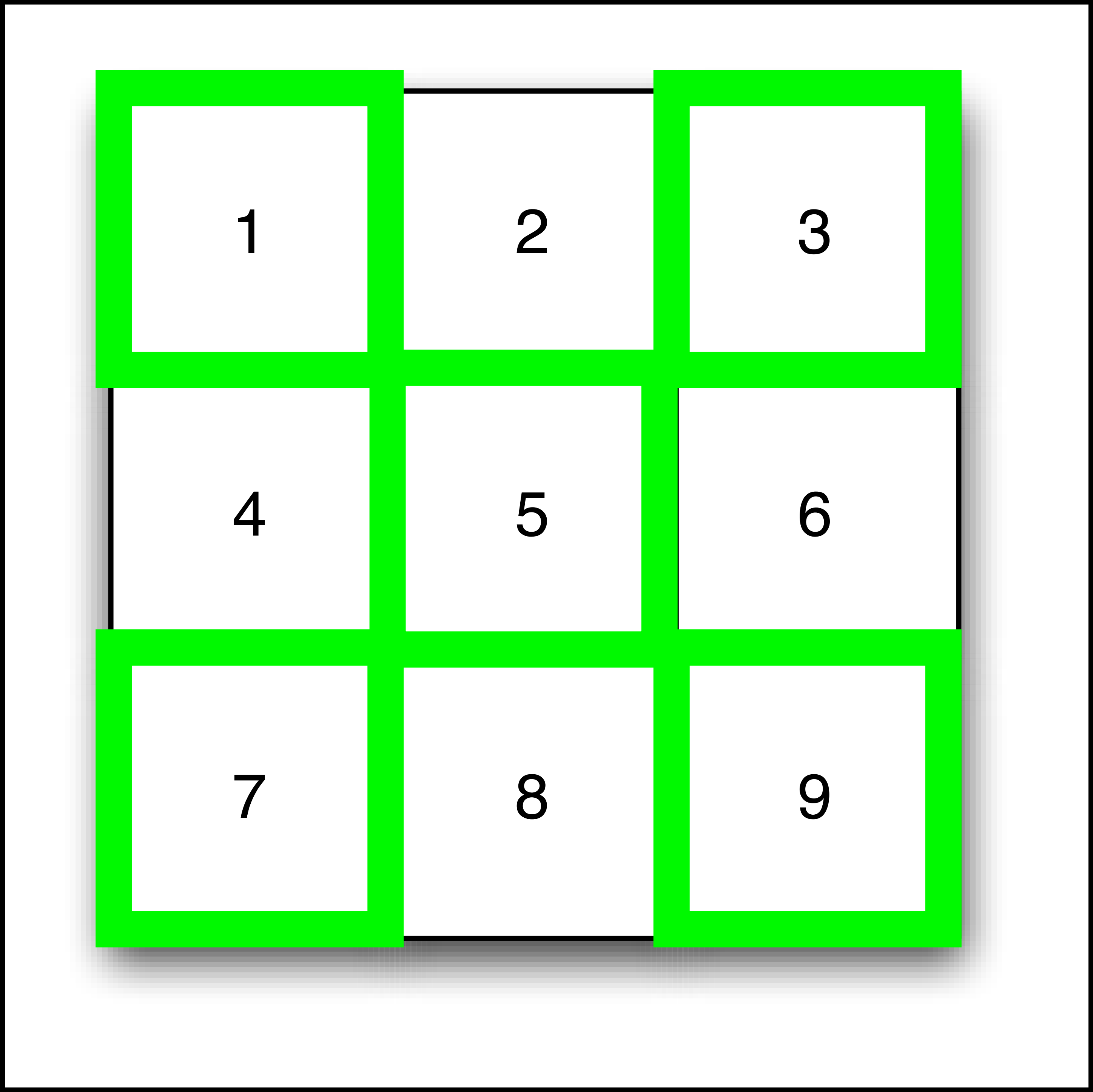

Bishop Weights

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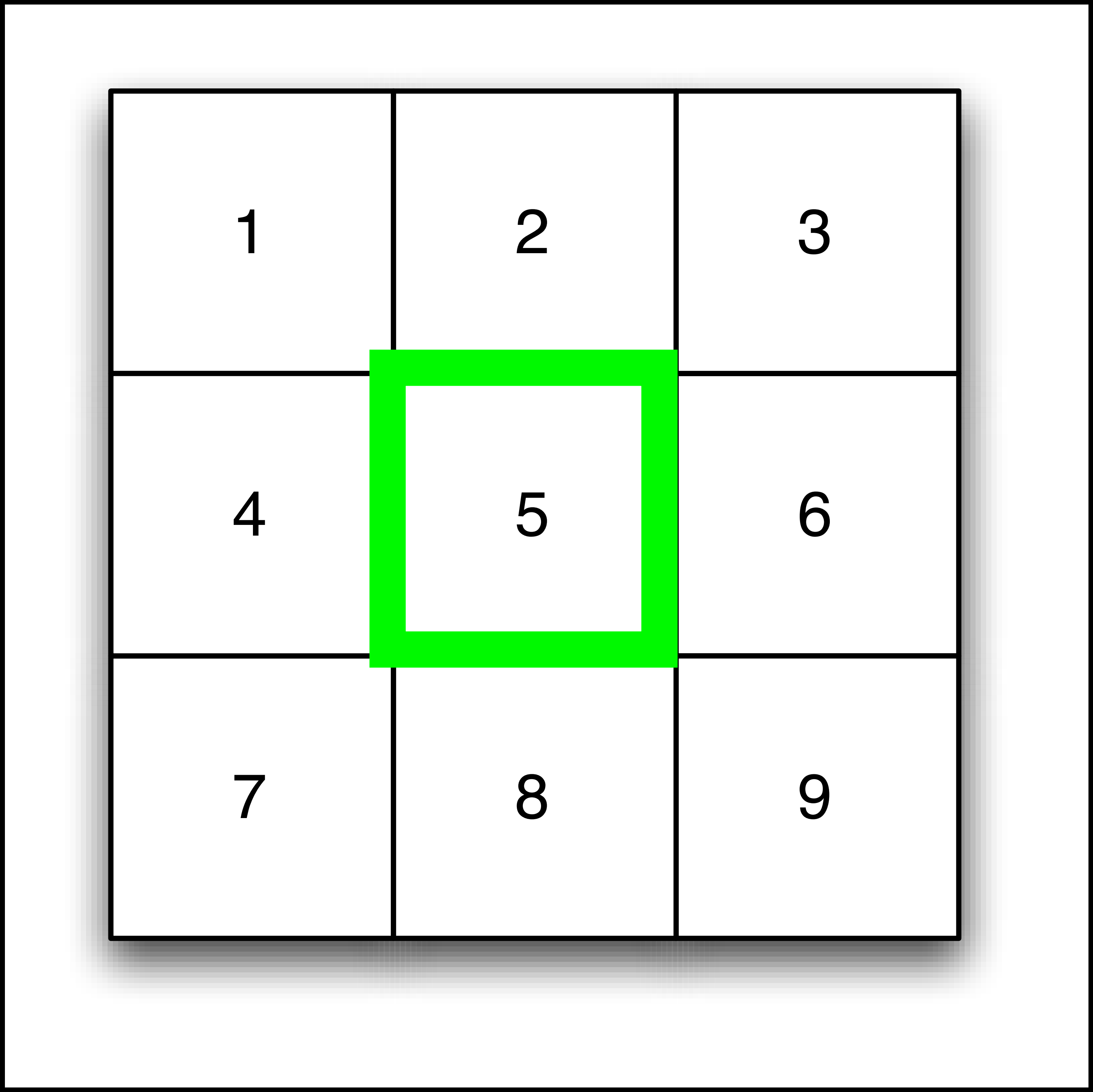
Neighbors for 5: 1, 3, 7, 9

Queen Contiguity



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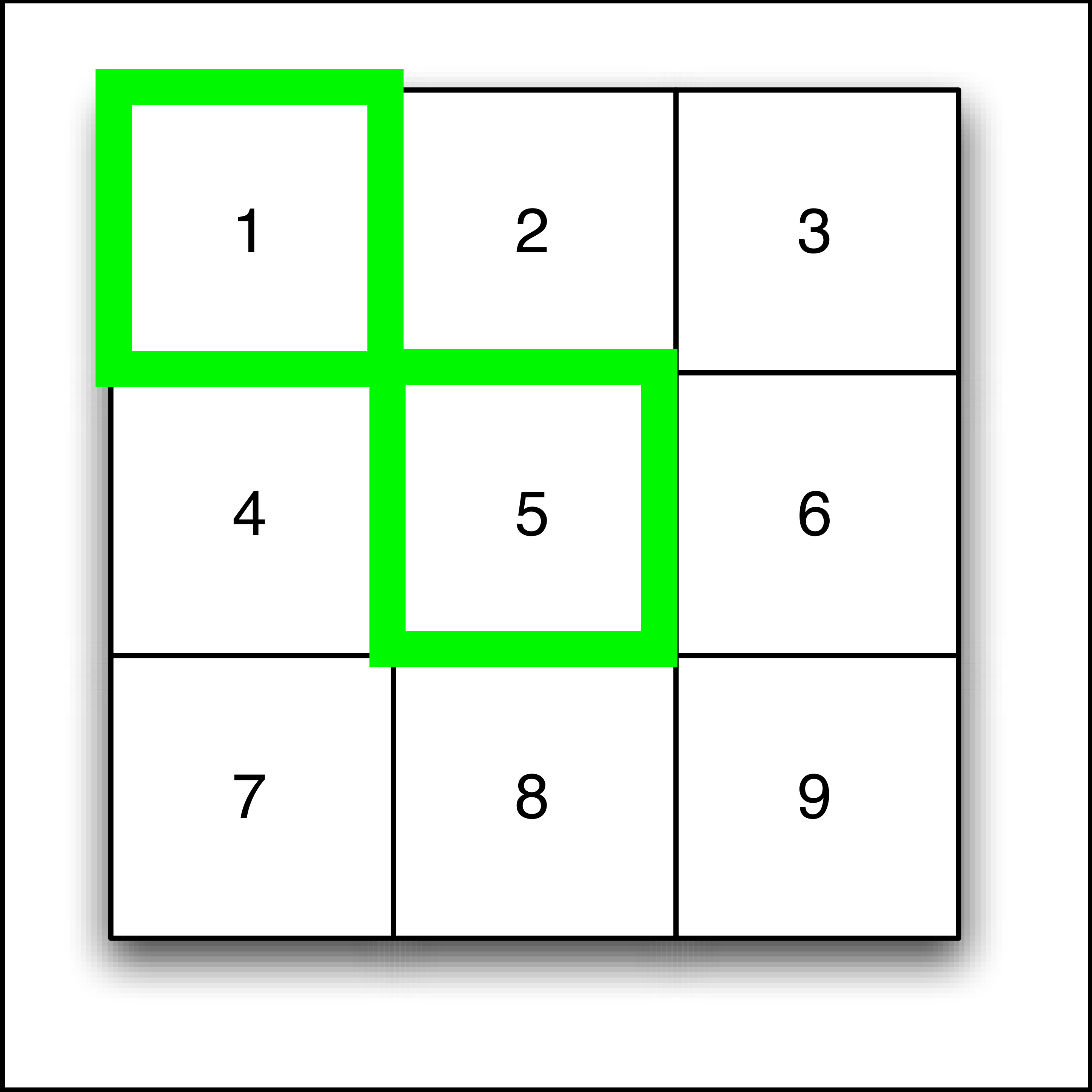

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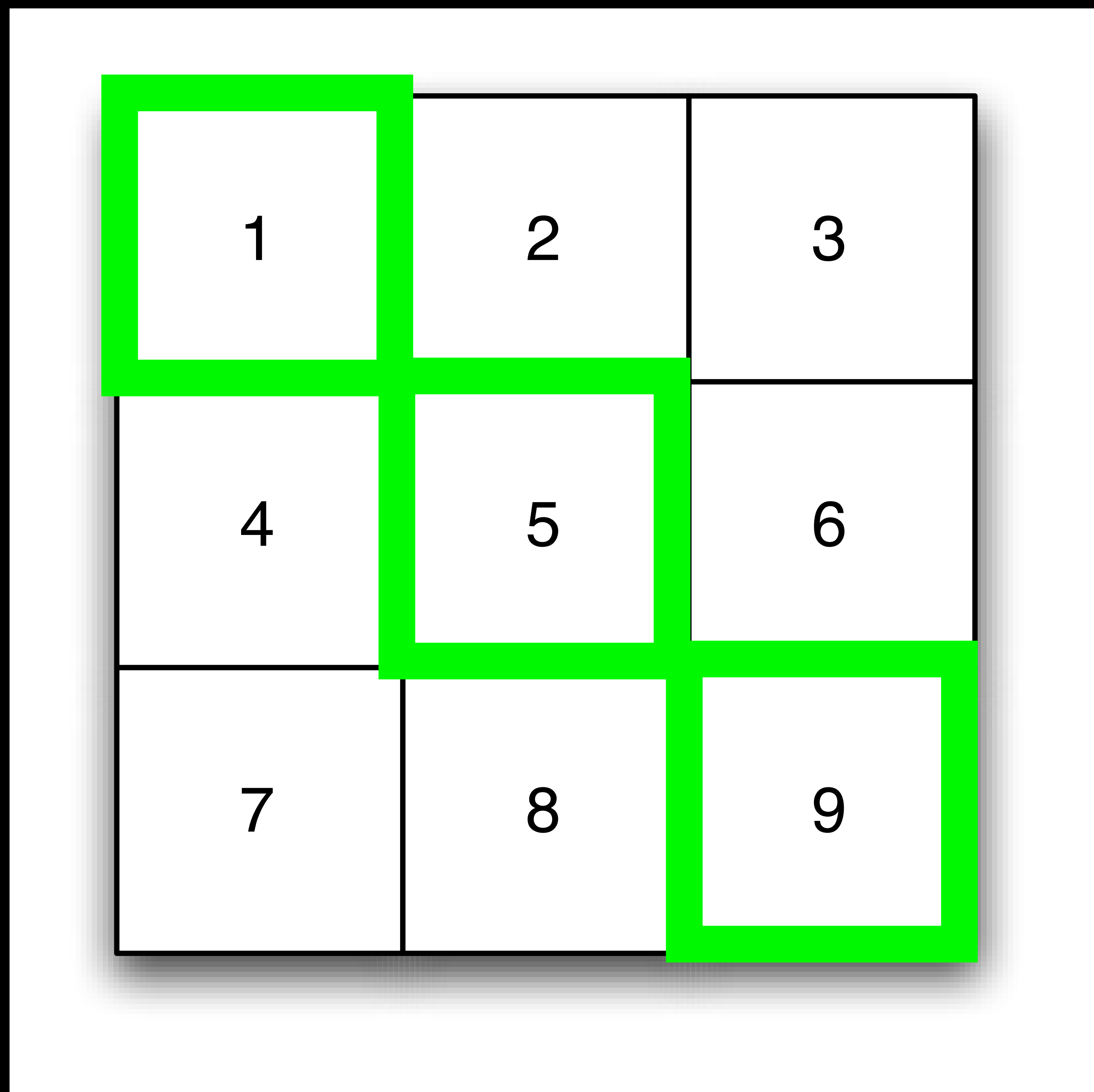

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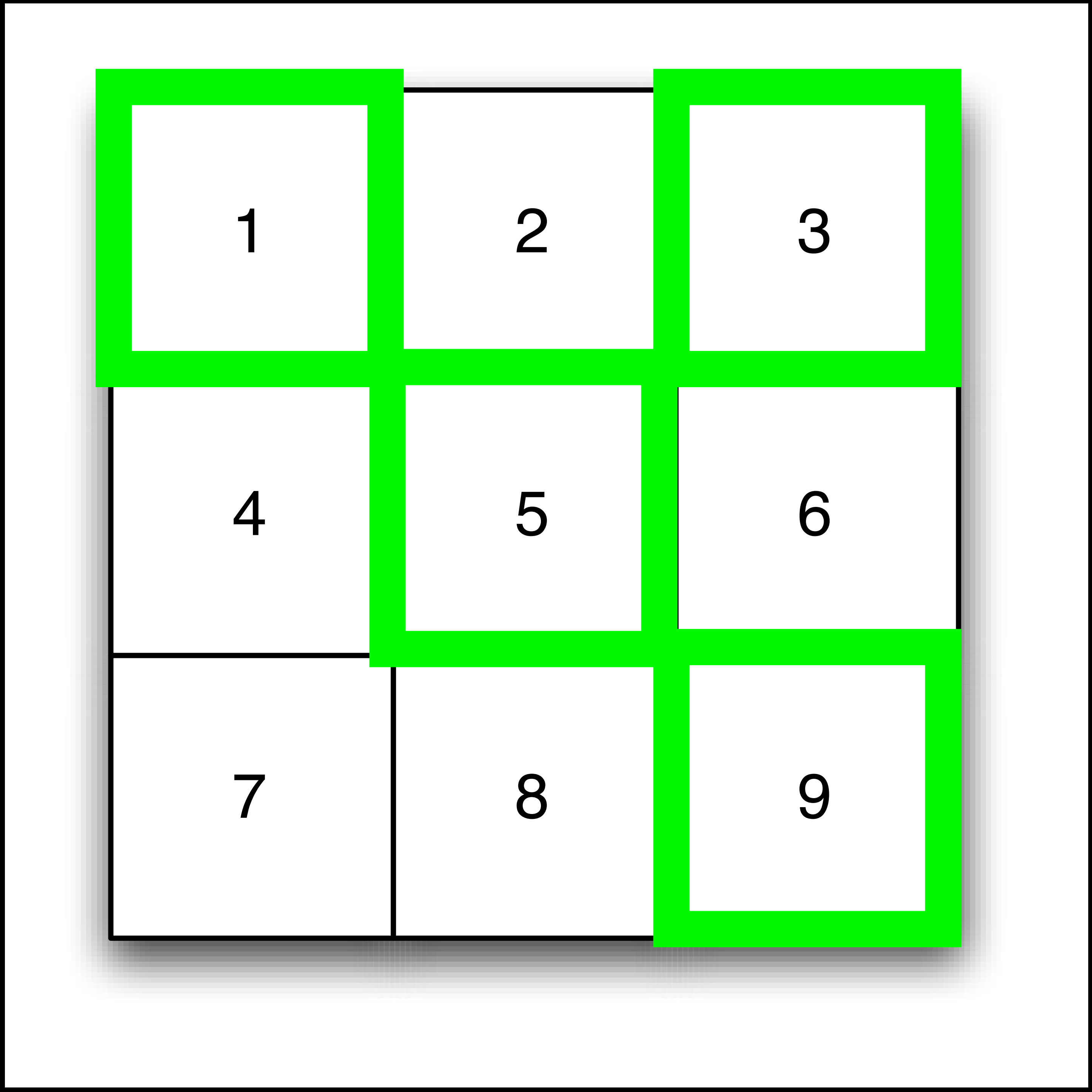

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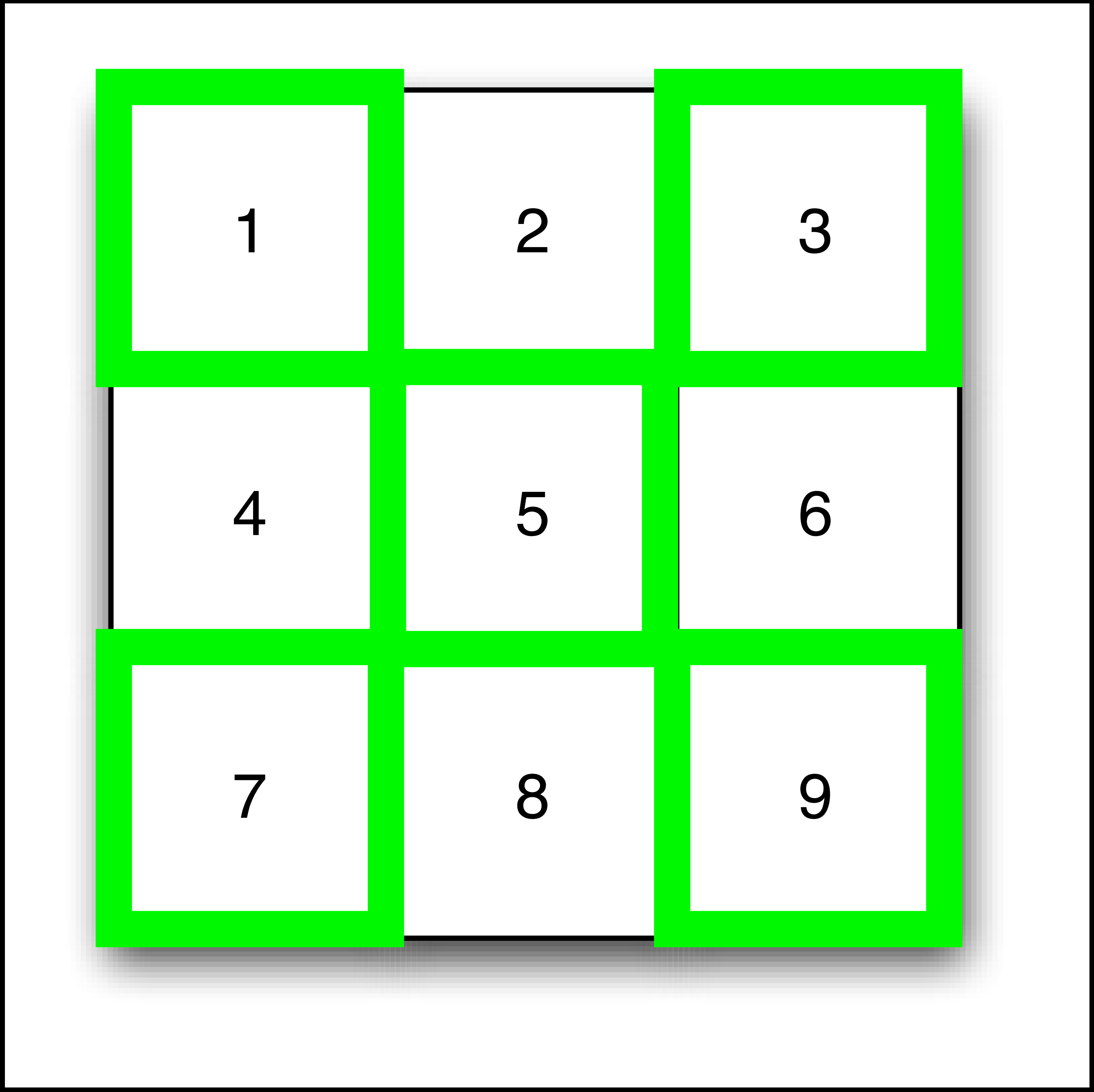

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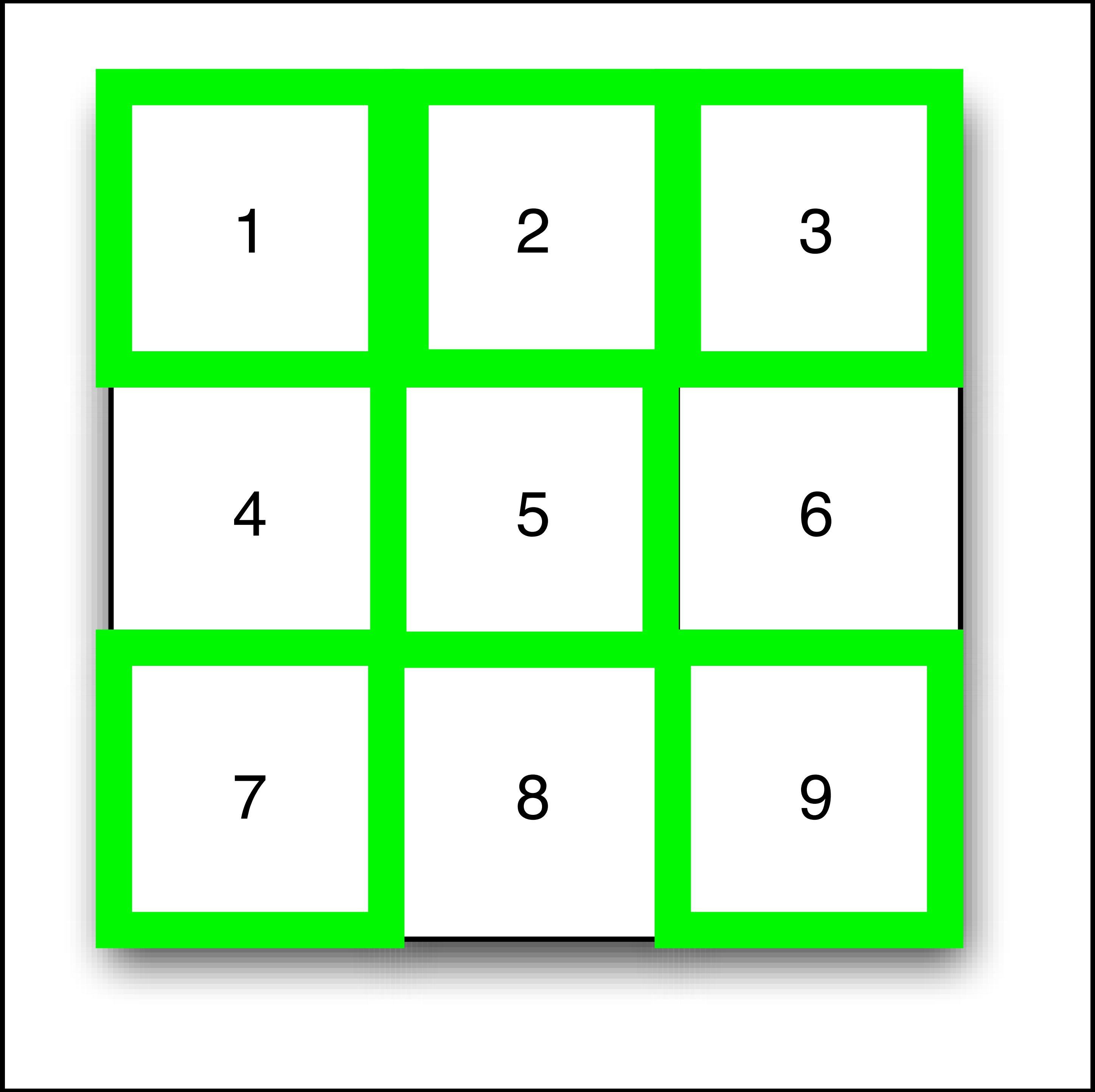

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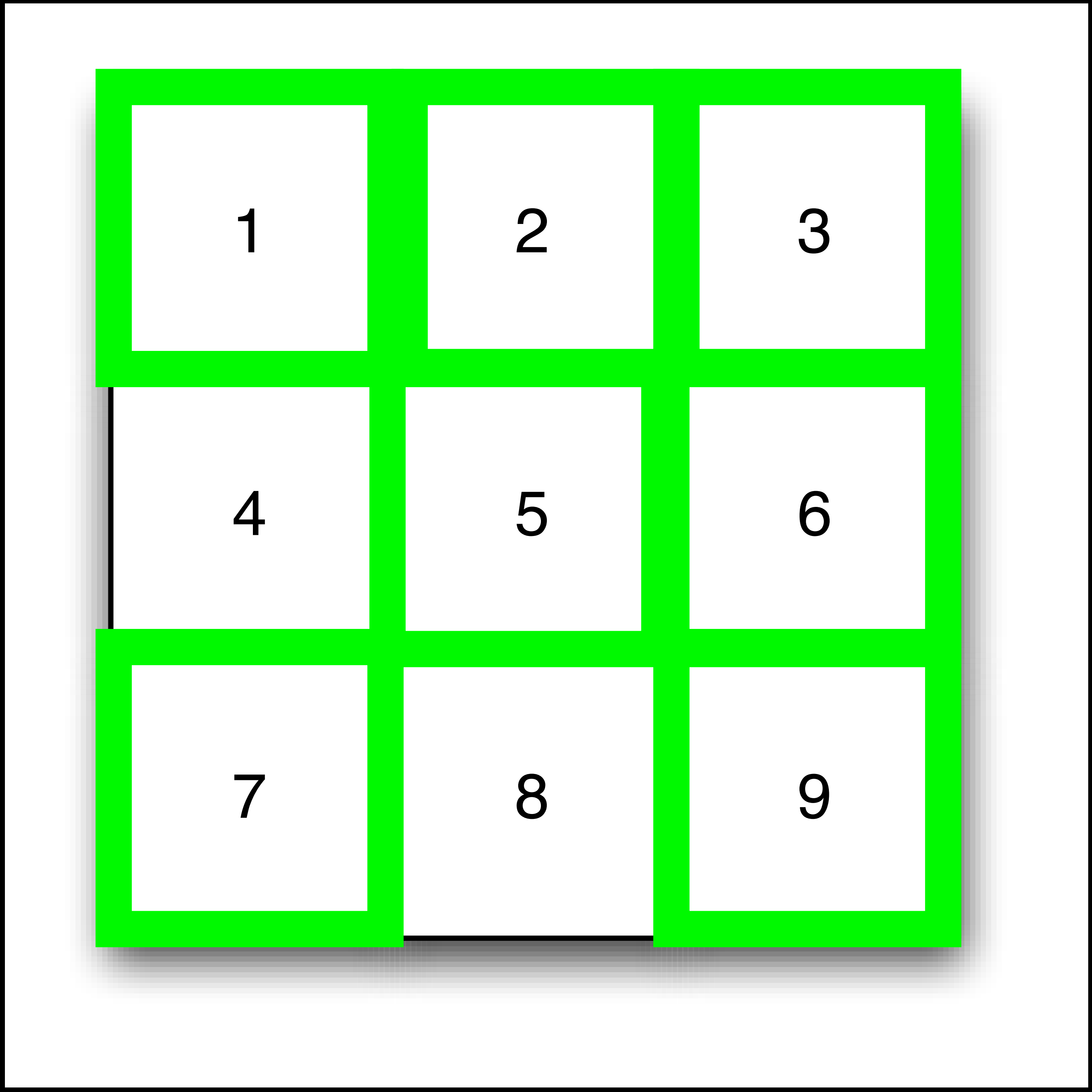

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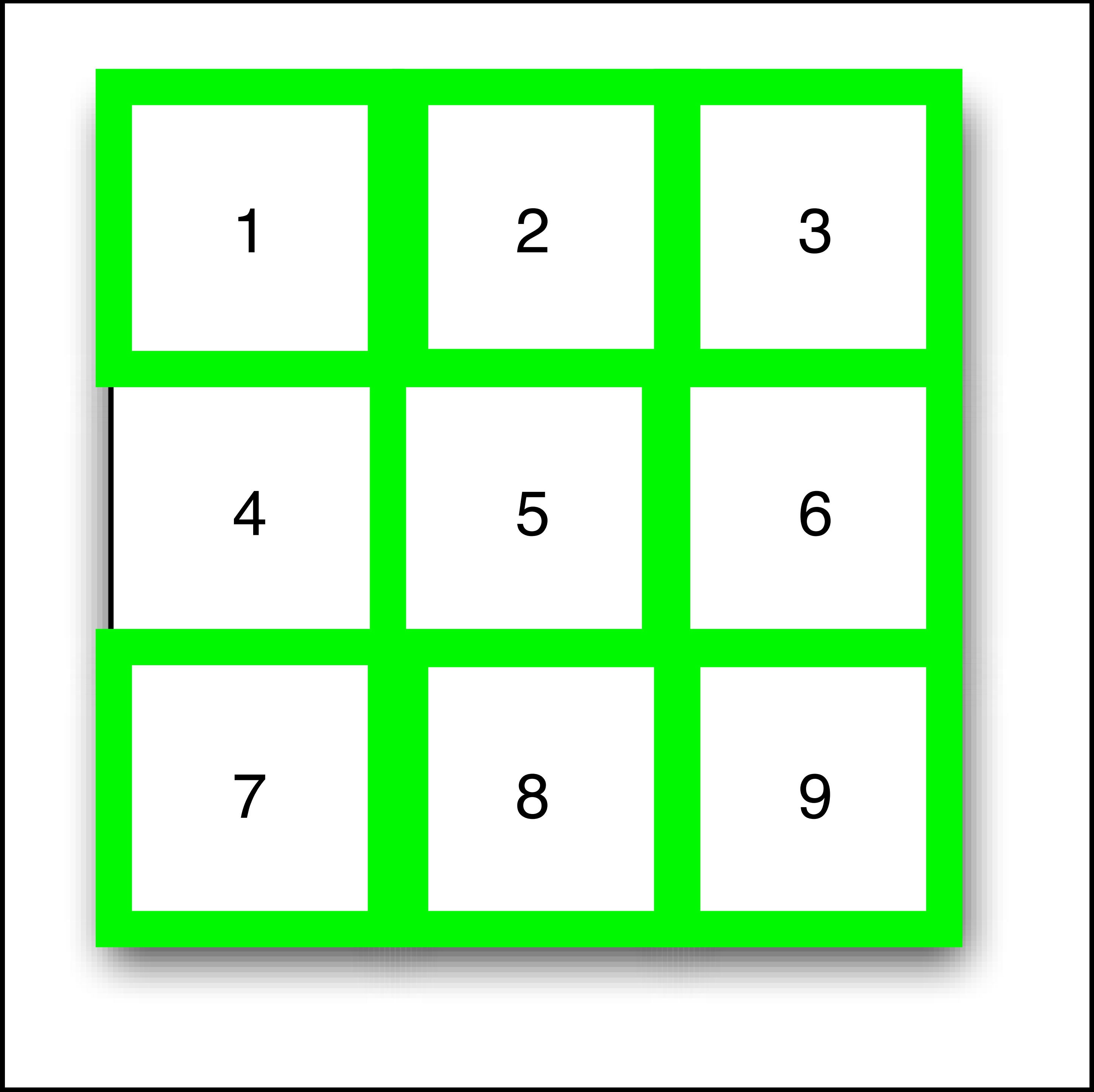

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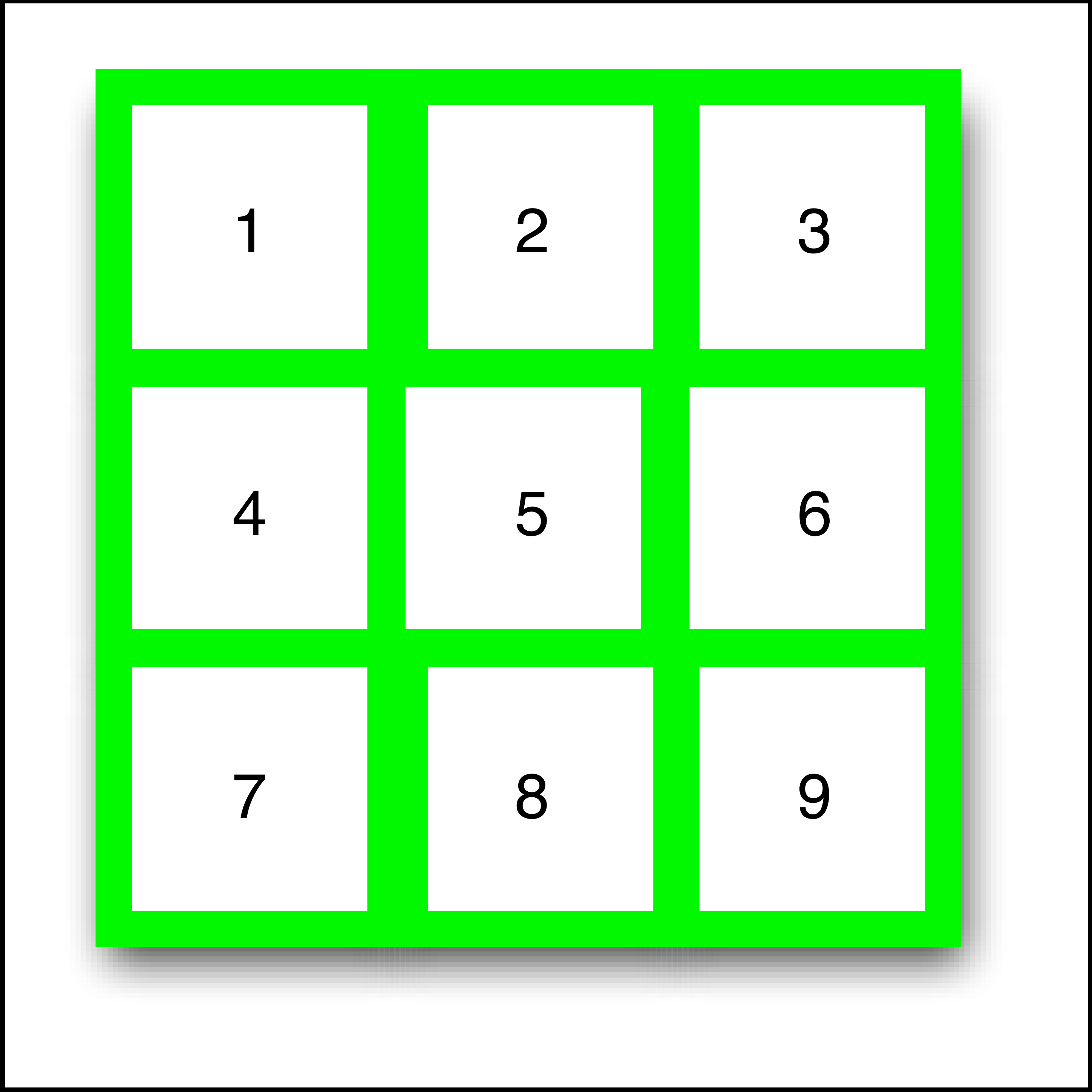

Queen Weights

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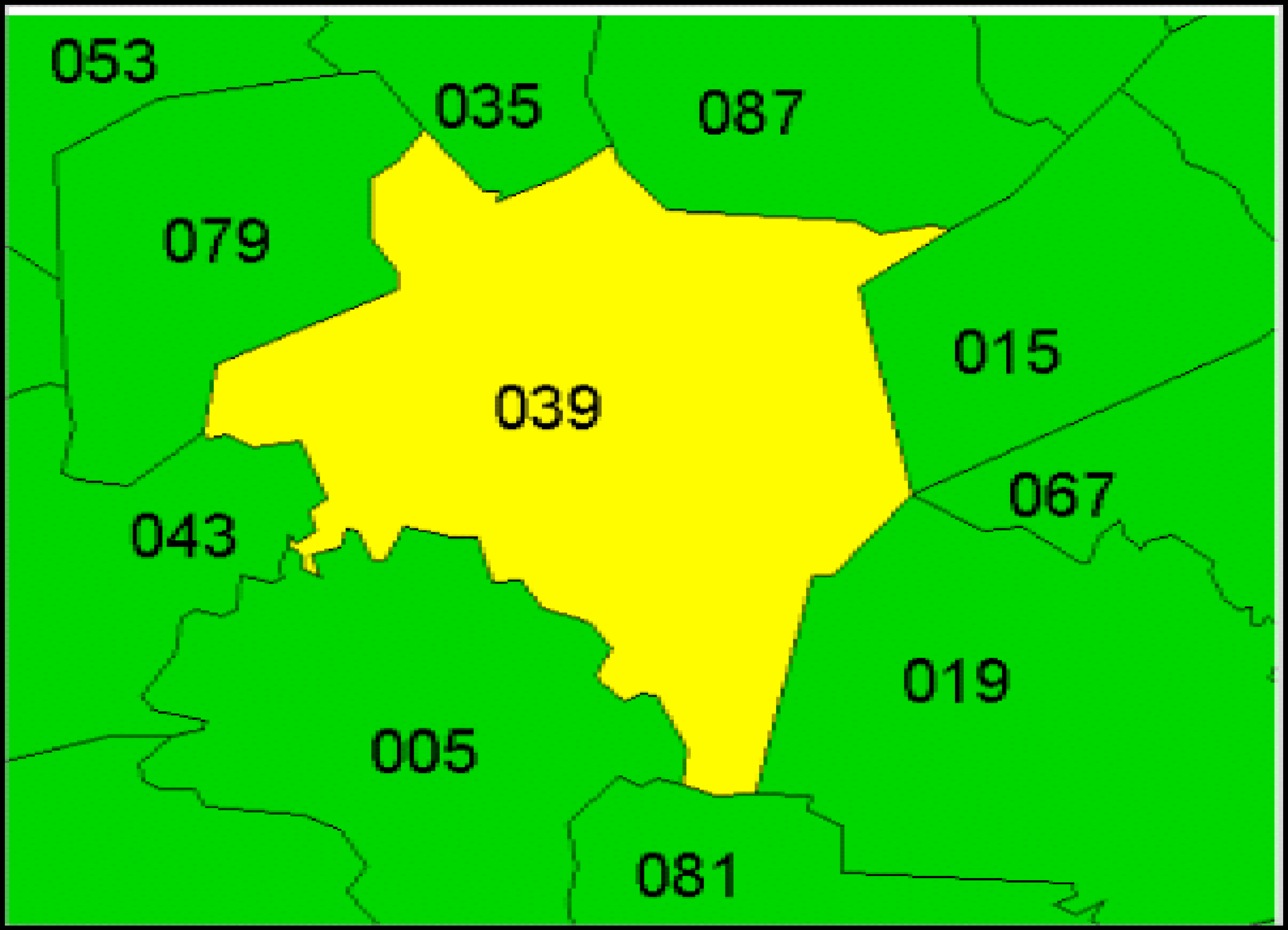
8 Neighbors for 5

Both Border and Vertex

Irregular Lattice Contiguity



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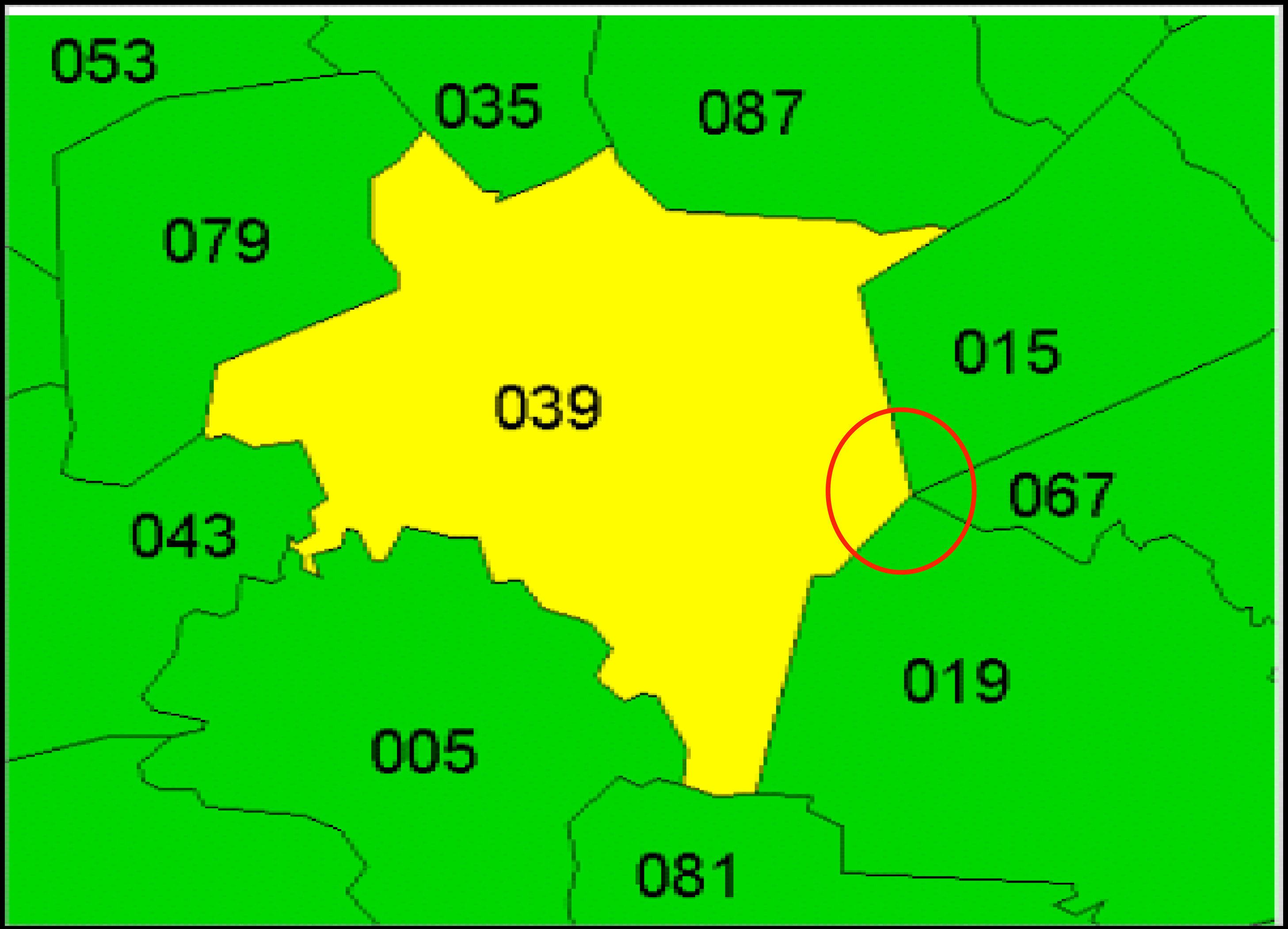

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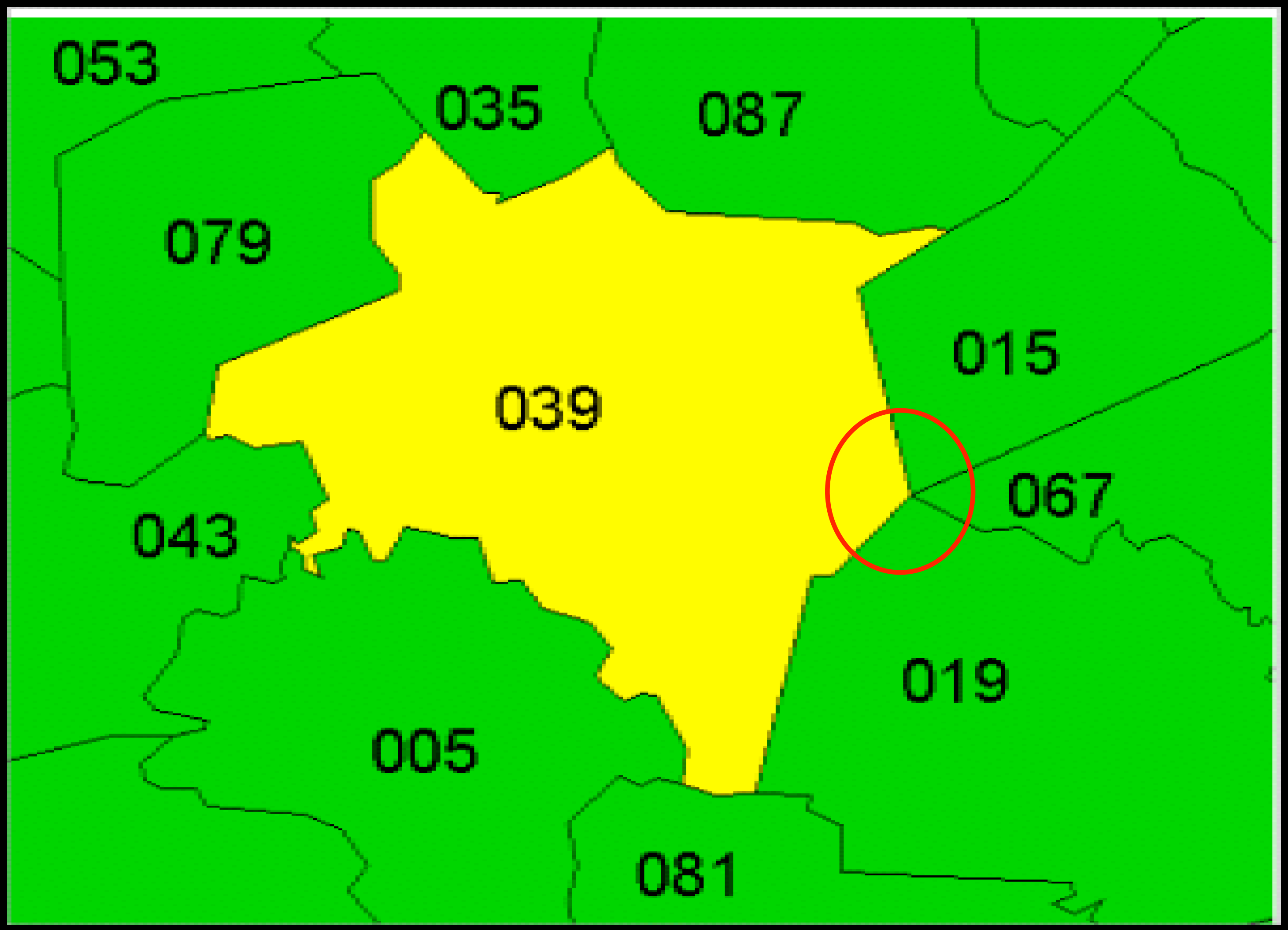
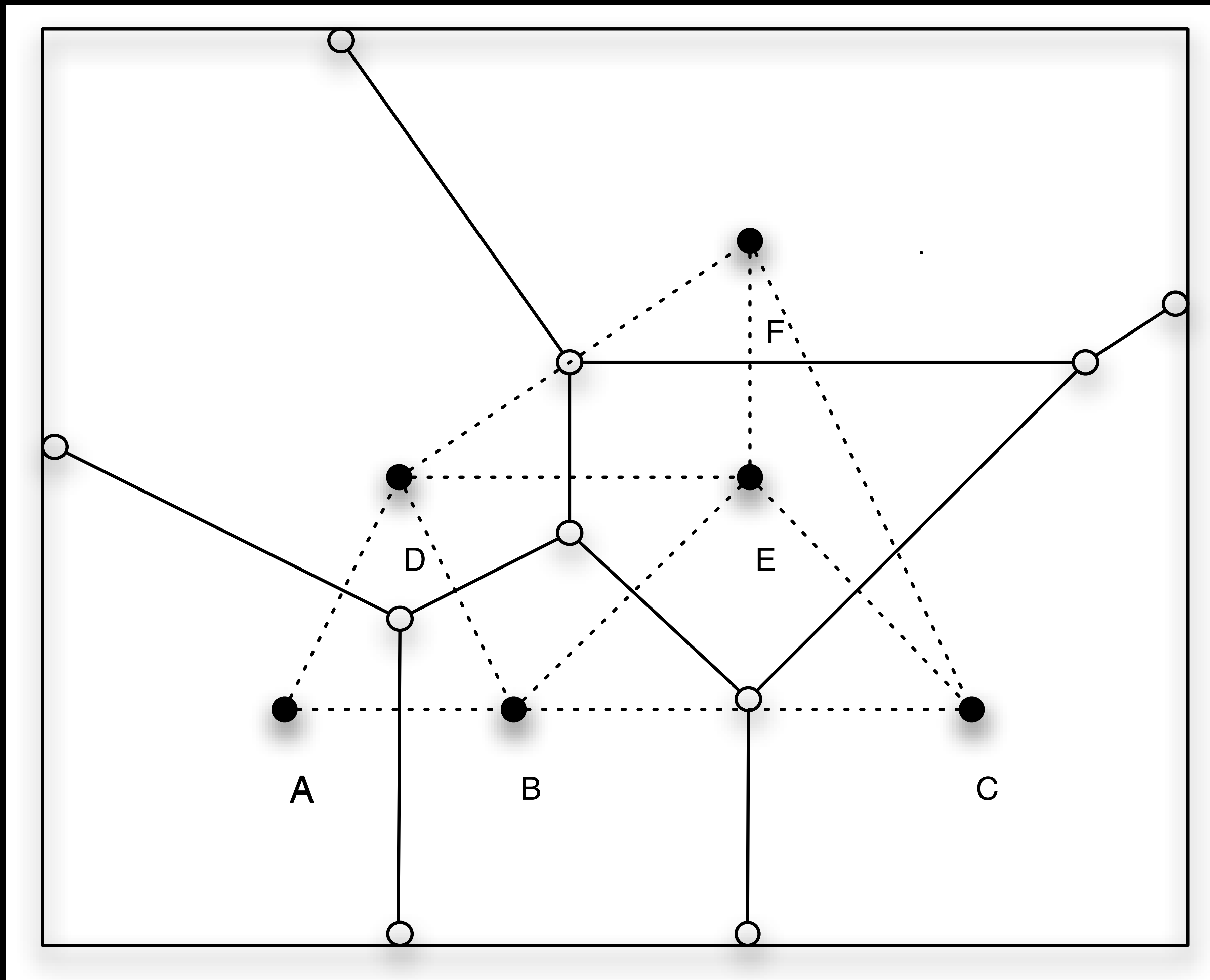


Rook Contiguity: common boarder only
Queen Contiguity: border and vertices - 039 and 067

Point Contiguity



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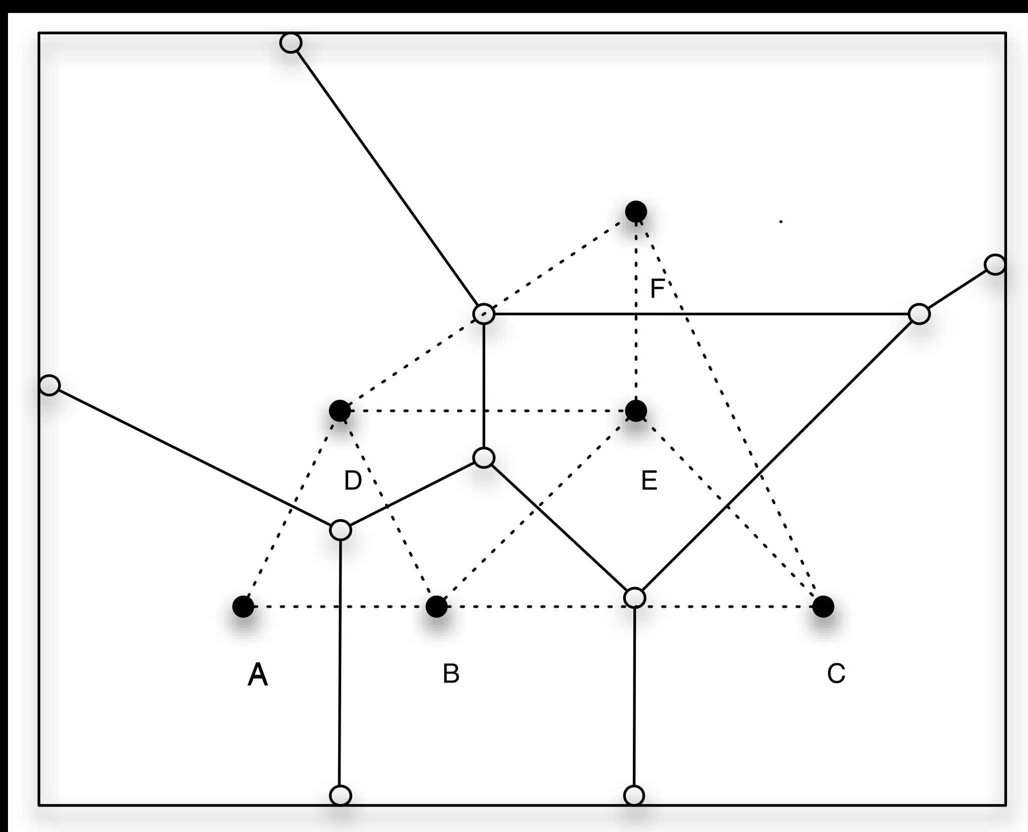

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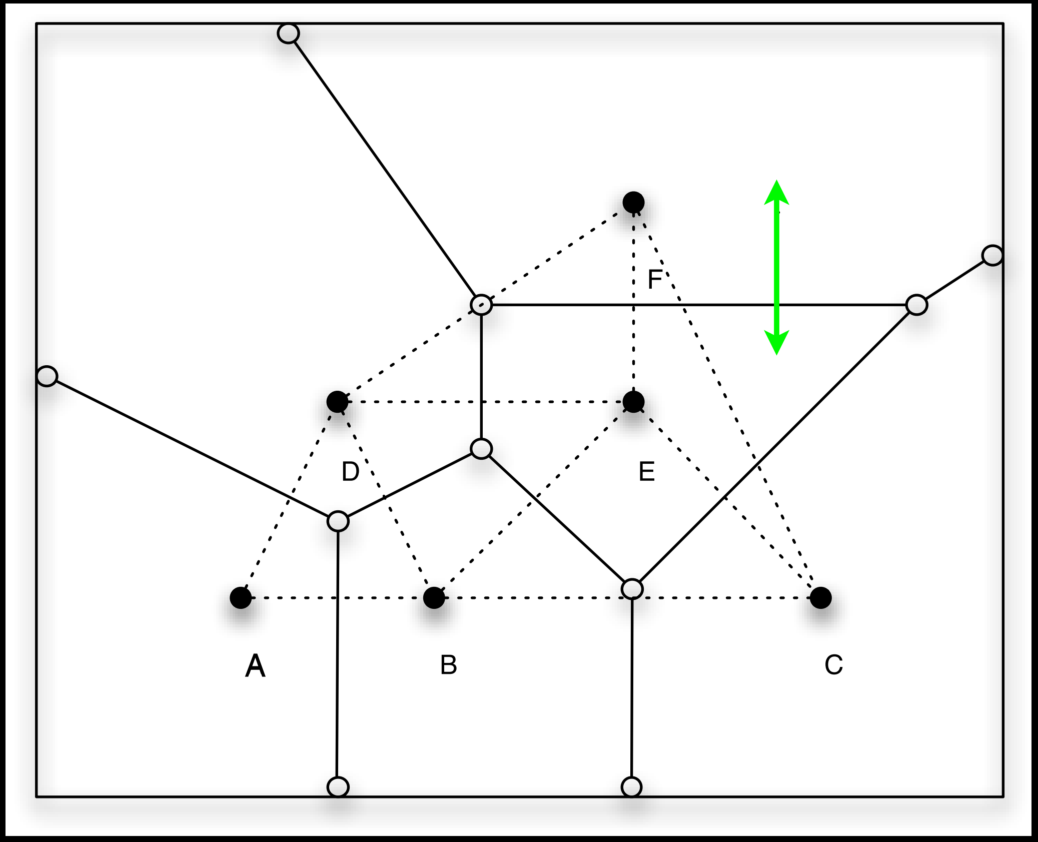

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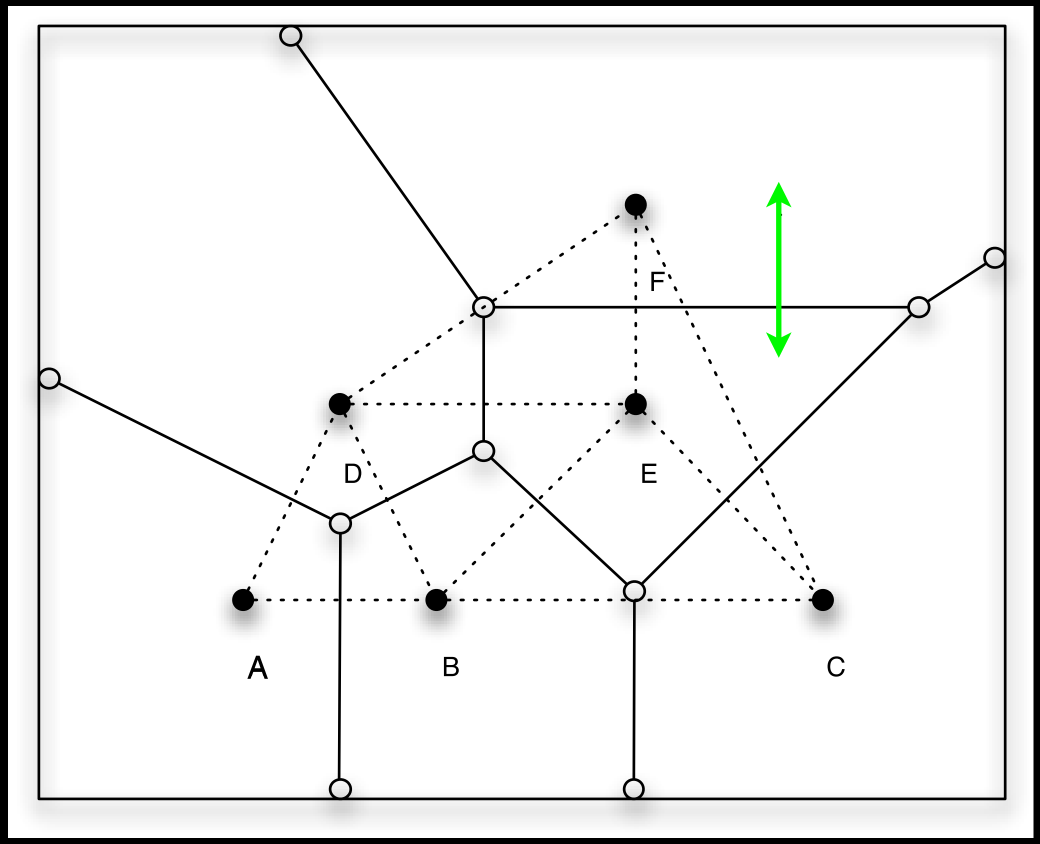

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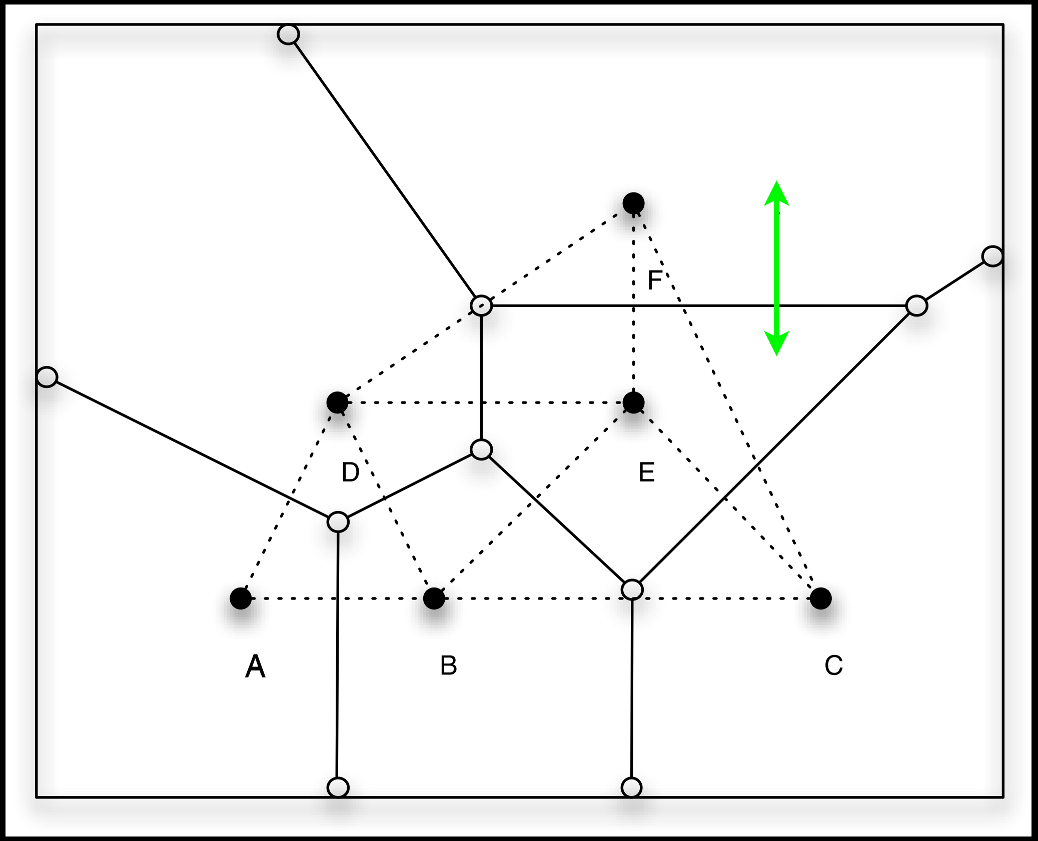

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Distance-Based Weights

Distance Measures

- Point (x_i, y_i)
- Interpoint Distances
- Metrics
  - Euclidean $d_{ij}^e = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$
  - Manhattan $d_{ij}^m = |x_i - x_j| + |y_i - y_j|$
  - Minkowski $d_{ij}^p = (|x_i - x_j|^p + |y_i - y_j|^p)^{(1/p)}$

Distance Measures

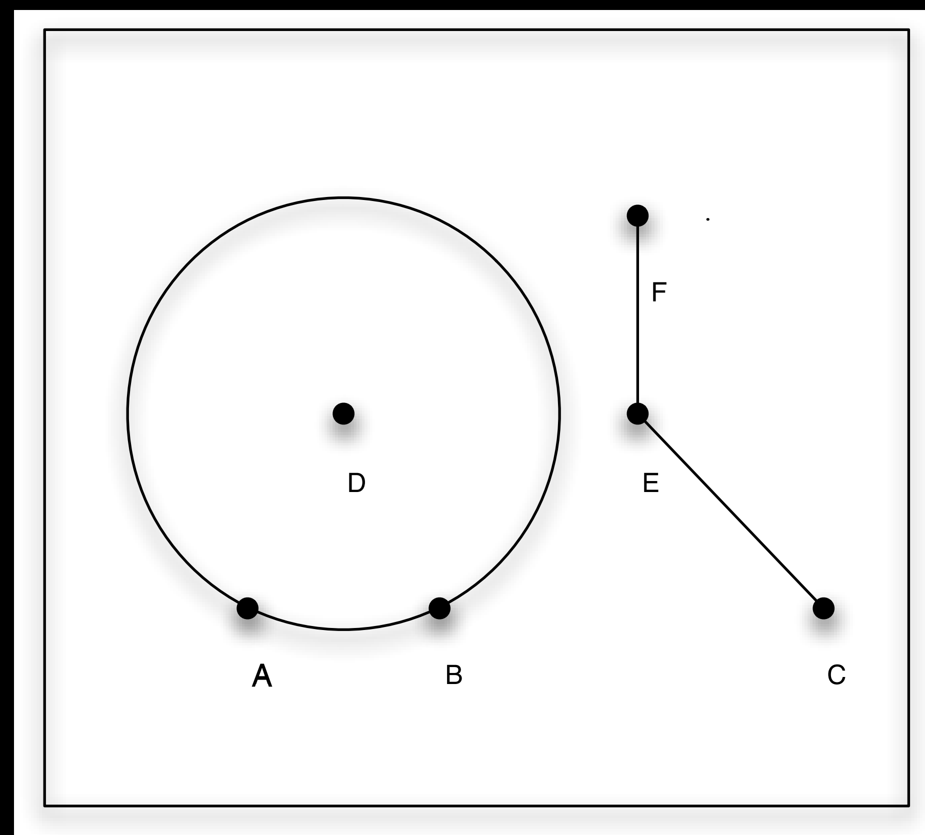
- Other
  - Road network
  - Actual travel
- Unprojected coordinates
  - Straight line distance measure inappropriate
  - Use great circle distance

Distance Measures

- Unprojected coordinates
  - Straight line distance measure inappropriate
  - Use great circle distance

$$d_{ij}^c = R.\arccos[\sin(\mathrm{lat}_i).\sin(\mathrm{lat}_j) + \cos(\mathrm{lon}_i).\cos(\mathrm{lon}_j).\cos(\mathrm{lon}_i - \mathrm{lon}_j)]$$

Interpoint Distance



$$A(10,10), B(20,10), C(40,10), D(15,20), E(30,20), F(30,30)$$

Interpoint Eucildean Distance

	B	C	D	E	F
A	10.0	30.0	11.2	22.4	28.3
B		20.0	11.2	14.1	22.4
C			26.9	14.1	22.4
D				15.0	18.0
E					10.0

Distance Bands

$w_{ij} = 1$ when $d_{ij} \leq \delta$, and $w_{ij} = 0$ otherwise

δ is a preset critical distance cutoff

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C is an island

Nearest Neighbor Distances

10 (A-B)
10 (B-A)
14.1 (C-E)
11.2 (D-A and D-B),
10 (E-F)
10 (F-E)

Take Maximum nn-distance

$$\delta = 14.1$$

Max nn-distance Band

$w_{ij} = 1$ when $d_{ij} \leq \delta$, and $w_{ij} = 0$ otherwise

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Max nn-distance Band

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C is connected

Max nn-distance Band

- Symmetric Matrix
- Avoids islands
- Too many neighbors for clustered locations
- Driven by maximum nn-distance

K Nearest Neighbors

- Avoids islands
- select k nearest neighbors for each location
- e.g., k=3

K Nearest Neighbors

- Avoids islands
- select k nearest neighbors for each location
- e.g., k=3

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Ties

$d_{FB} = d_{FC} = 22.4$

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

	B	C	D	E	F
A	10.0	30.0	11.2	22.4	28.3
B		20.0	11.2	14.1	22.4
C			26.9	14.1	22.4
D				15.0	18.0
E					10.0

Ties

$d_{FB} = d_{FC} = 22.4$

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

	B	C	D	E	F
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E					10.0

Ties

$d_{FB} = d_{FC} = 22.4$

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E					10.0

Ties

$d_{FB} = d_{FC} = 22.4$

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

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Ties

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Not symmetric

Ties

$d_{FB} = d_{FC} = 22.4$

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Ties

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C			26.9	14.1	22.4
D				15.0	18.0
E					10.0

Not symmetric

B is a 3nn to F

F is not a 3nn to B

Ties

$d_{FB} = d_{FC} = 22.4$

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

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C			26.9	14.1	22.4
D				15.0	18.0
E					10.0

Not symmetric

B is a 3nn to F

F is not a 3nn to B

Representative Points

- Represent a polygon by a point
- Measure distance between representative points
- Construct distance based W

Representative Points

- Capital, County Seat
- Central Point
- Polygon Centroid
- Minimum Bounding Rectangle Center

Centroid

- Polygon $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$

$$(x_n, y_n) = (x_1, y_1)$$

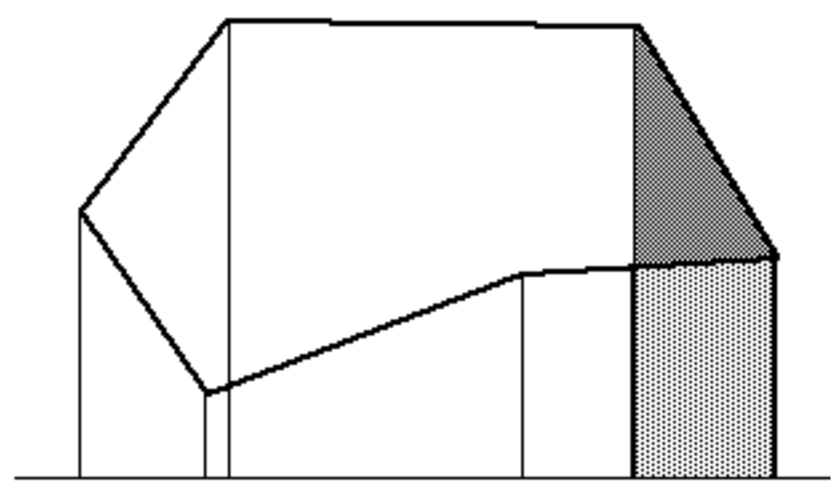
$$x_c = (1/6A) \sum_{i=1}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$y_c = (1/6A) \sum_{i=1}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

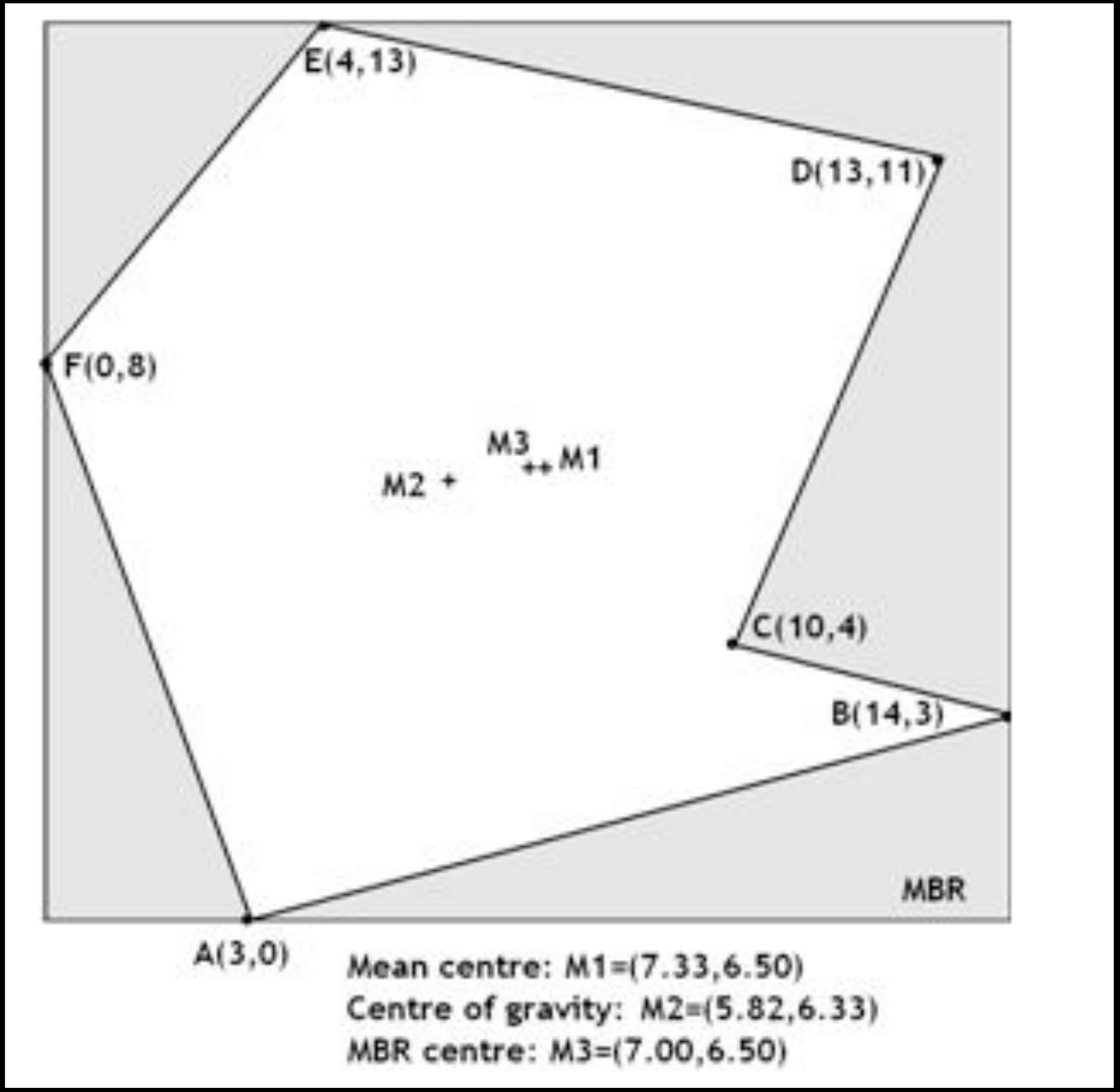
$$A = (1/2) \sum_{i=1}^{n-1} x_i y_{i+1} - x_{i+1} y_i$$

Polygon Area

- Negative if digitized ccw
- Trapezoids



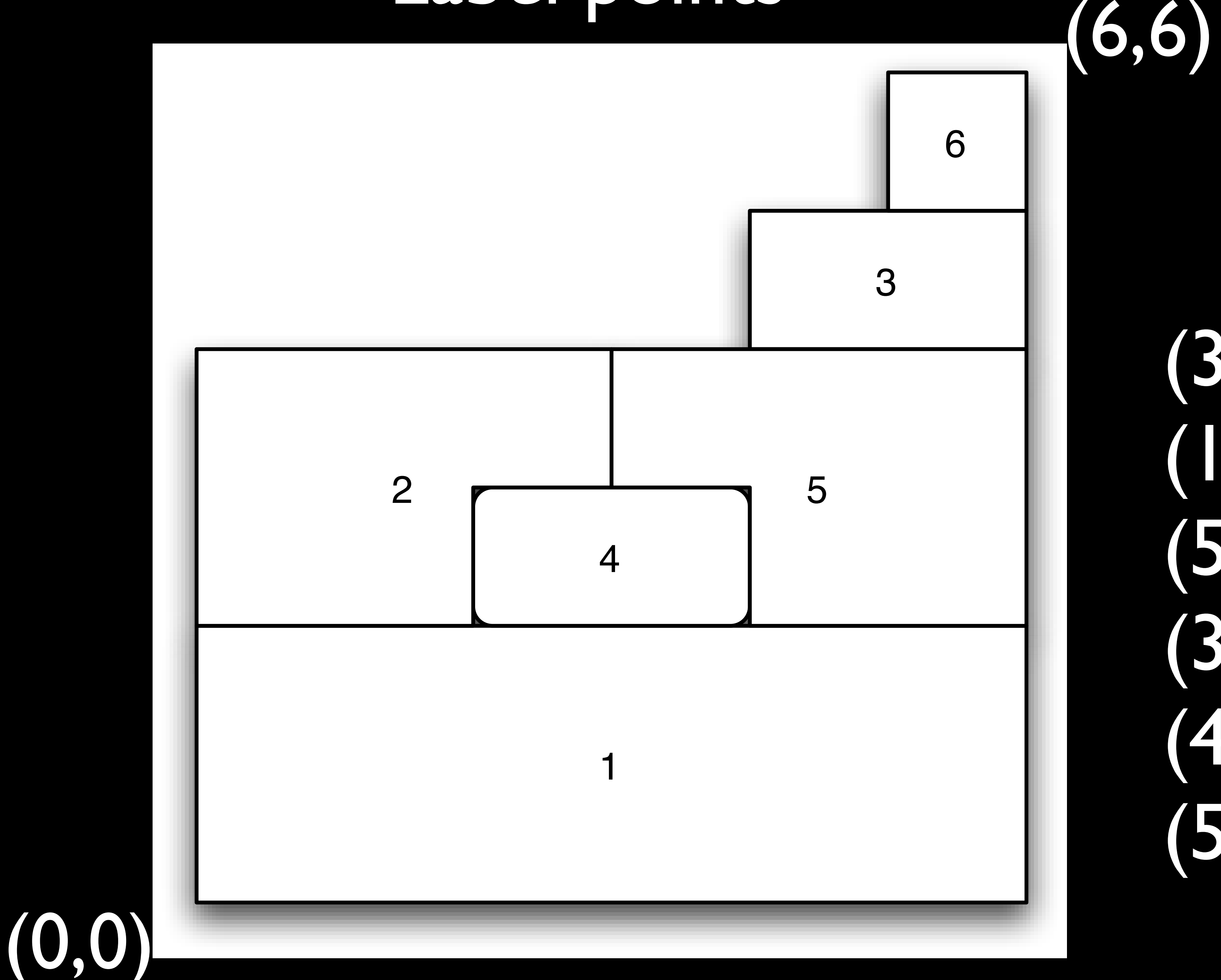
Representative Points



Source: http://www.spatialanalysisonline.com

Representative Points

Label points



(3,1)
(1.5,3)
(5, 4.5)
(3,2.5)
(4.5,3)
(5.5,5.5)

Distance Band

	2	3	4	5	6
1	2.5	4.0	1.5	2.5	5.1
2		3.8	1.6	3.0	4.7
3			2.8	1.6	1.1
4				1.6	3.9
5					2.7

$$\delta = 2.5$$

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Distance Band

	2	3	4	5	6
1	2.5	4.0	1.5	2.5	5.1
2		3.8	1.6	3.0	4.7
3			2.8	1.6	1.1
4				1.6	3.9
5					2.7

$$\delta = 2.5$$

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ \boxed{0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0} \end{bmatrix}$$

Distance Band

	2	3	4	5	6
1	2.5	4.0	1.5	2.5	5.1
2		3.8	1.6	3.0	4.7
3			2.8	1.6	1.1
4				1.6	3.9
5					2.7

$$\delta = 2.5$$

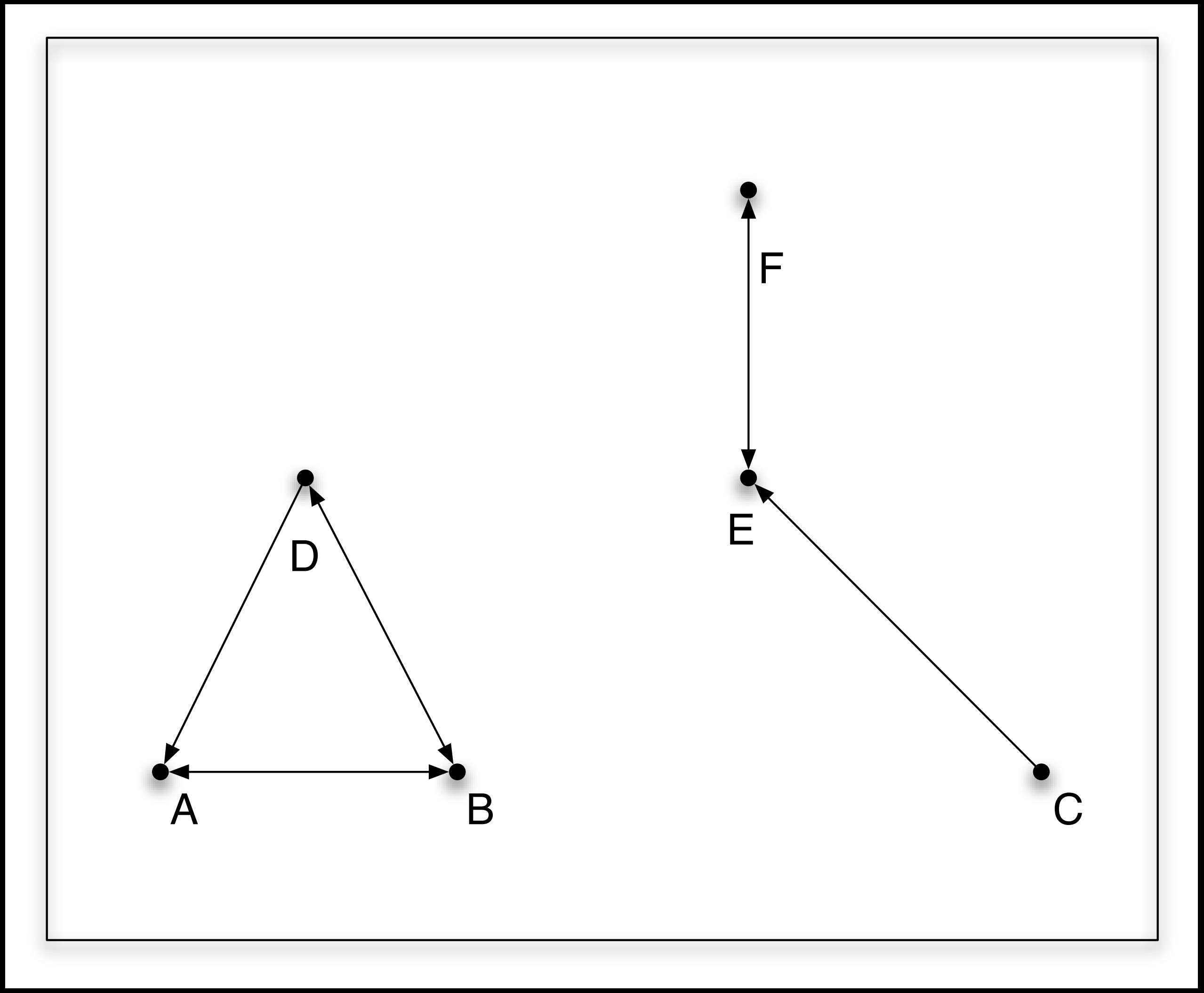
Variance in area leads to variance in connectivity

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Graph-Based Weights

- Points = Vertices
- Edges = Contiguity
- Definition of Edges based on Distance Criterion

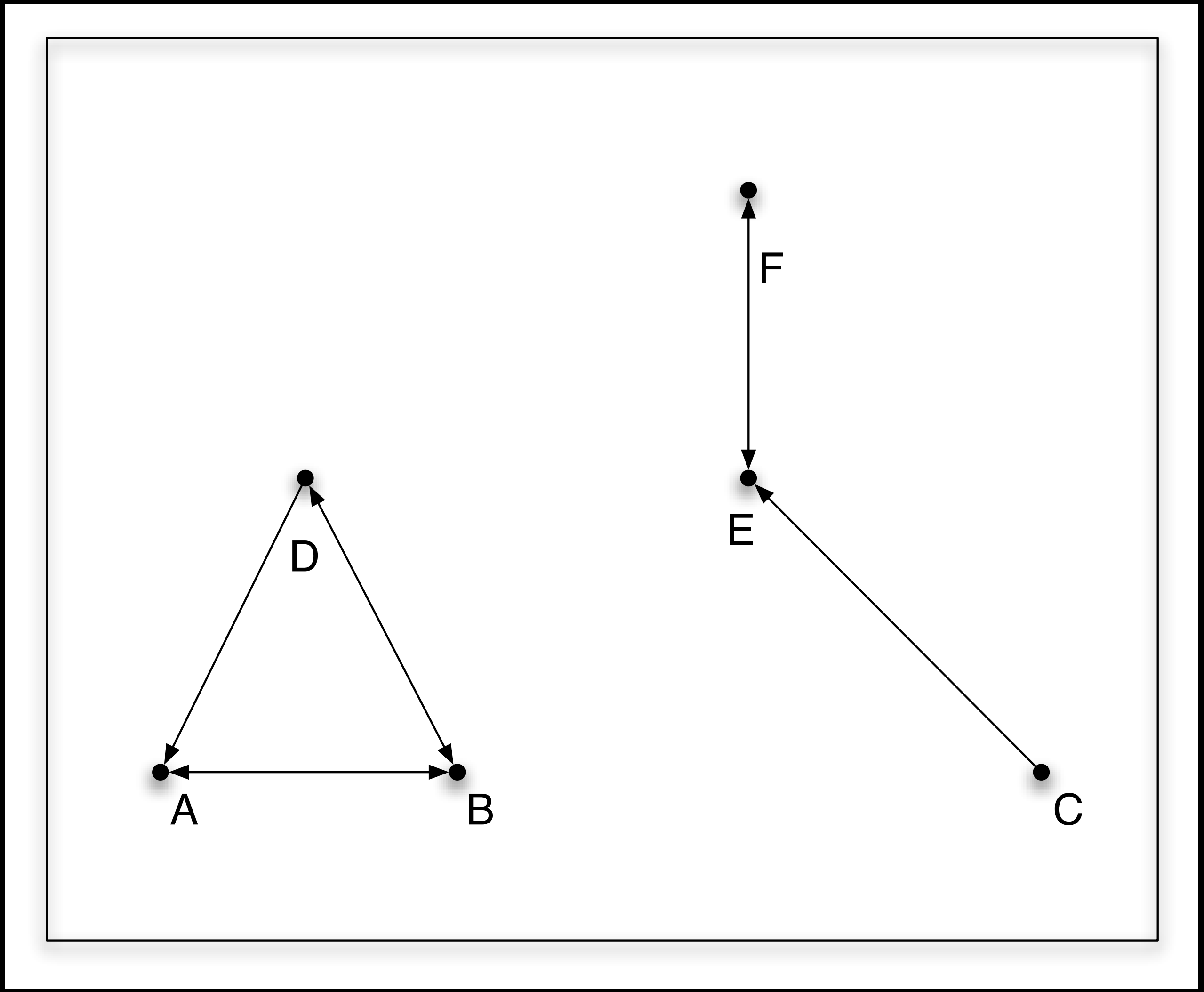
Nearest Neighbor Graph



Directed Graph

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

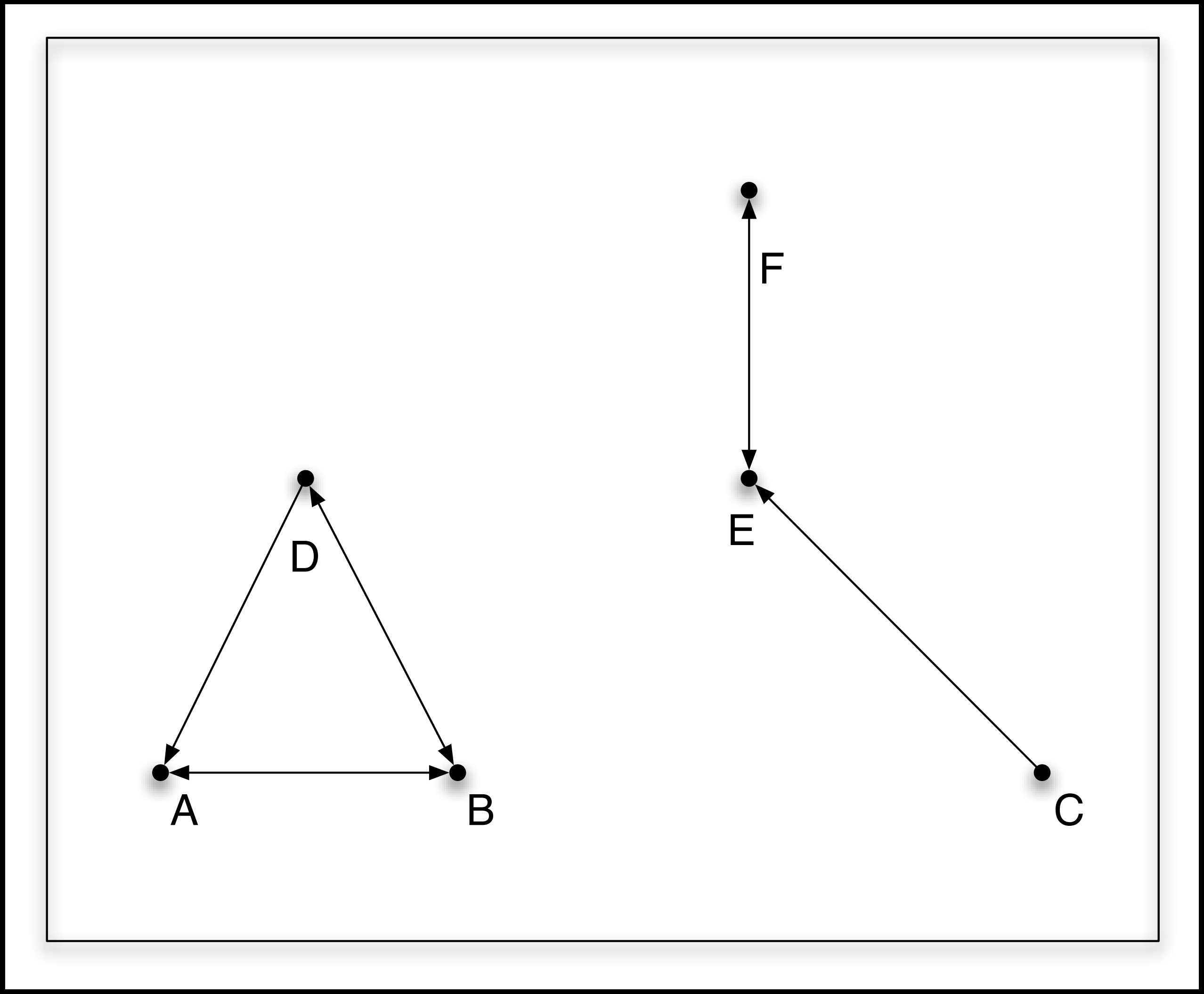
Nearest Neighbor Graph



Directed Graph

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

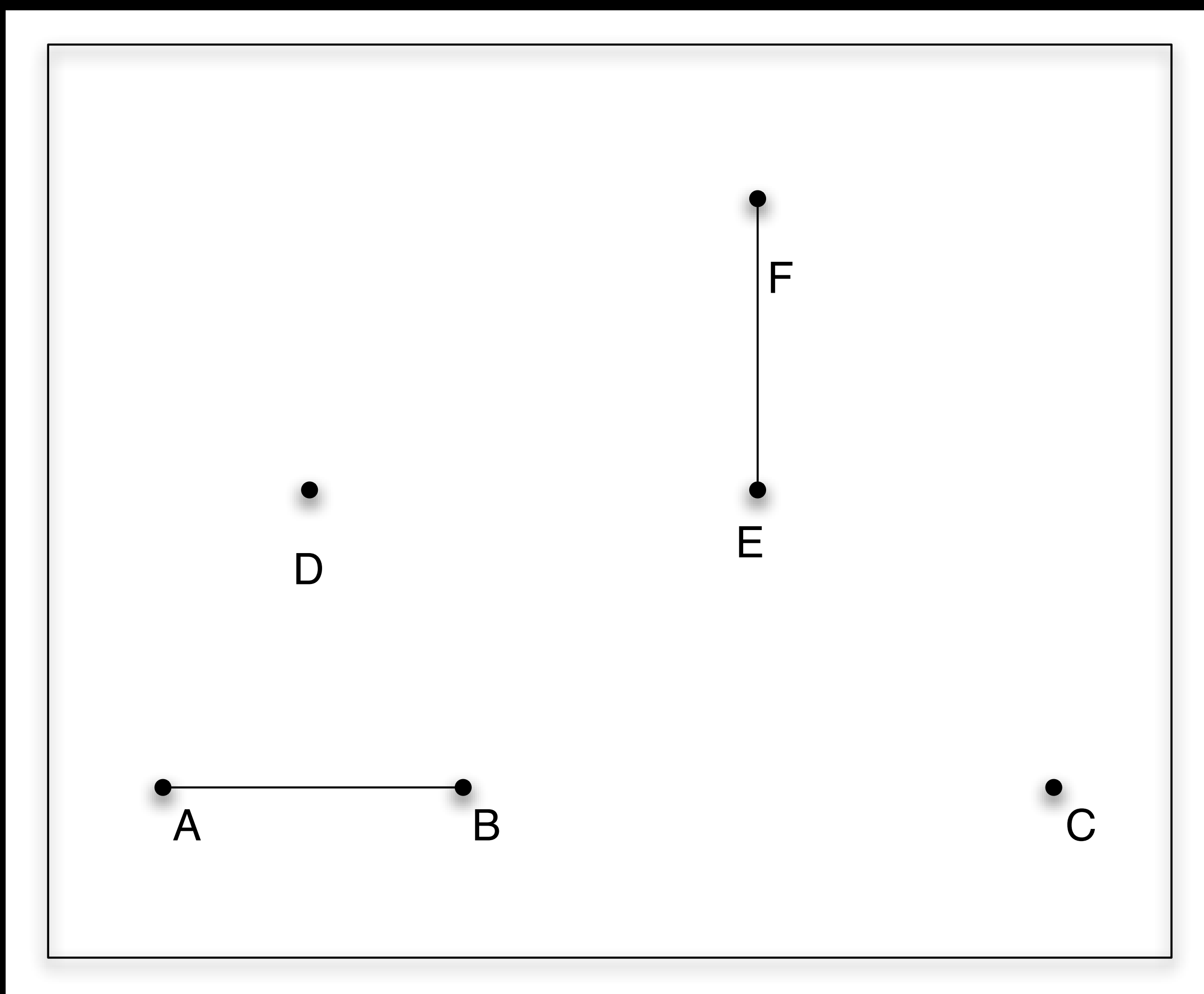
Nearest Neighbor Graph



Directed Graph

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Mutual Nearest Neighbors

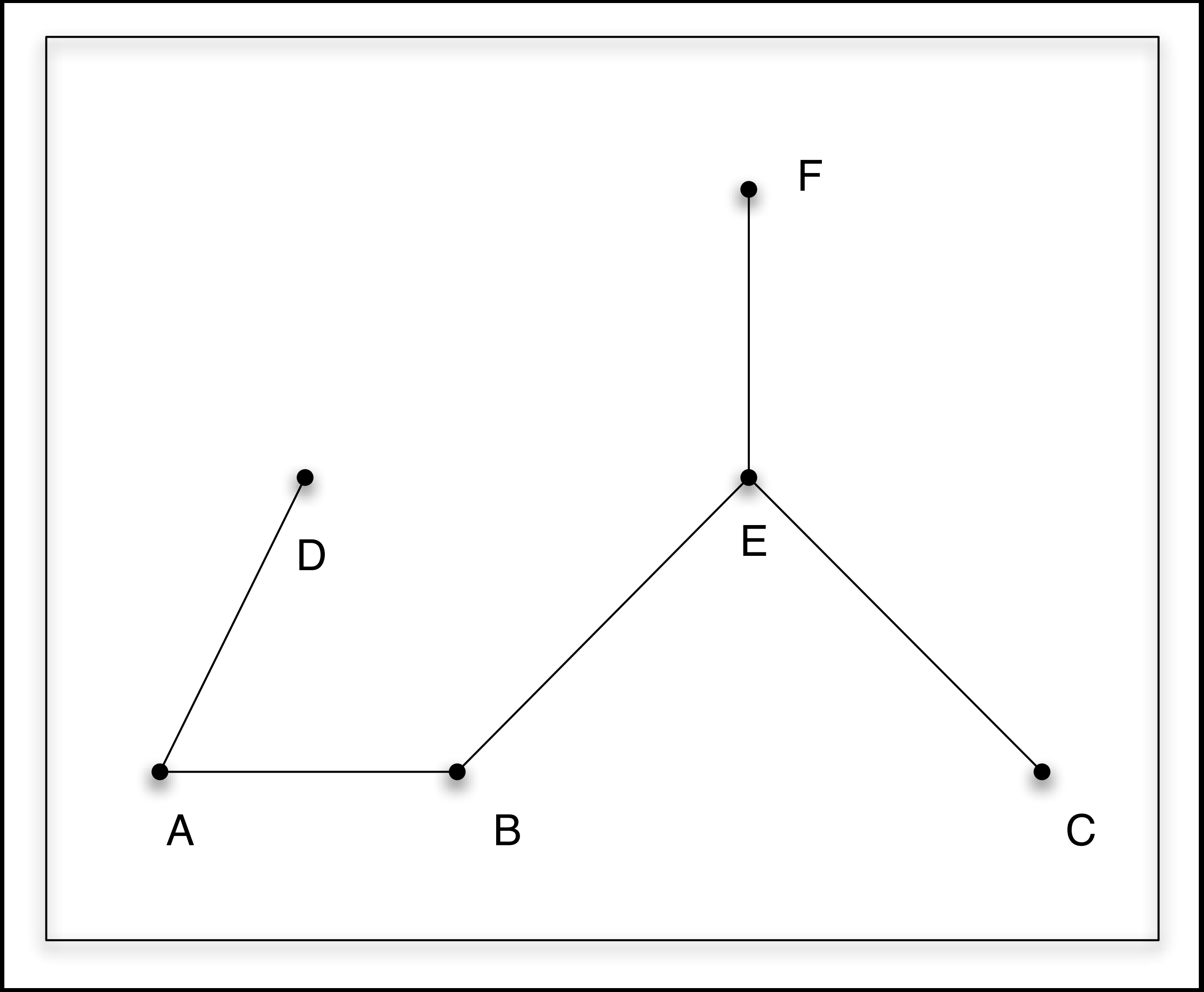


$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Sparse

Non-directed Graph

Minimum Spanning Tree

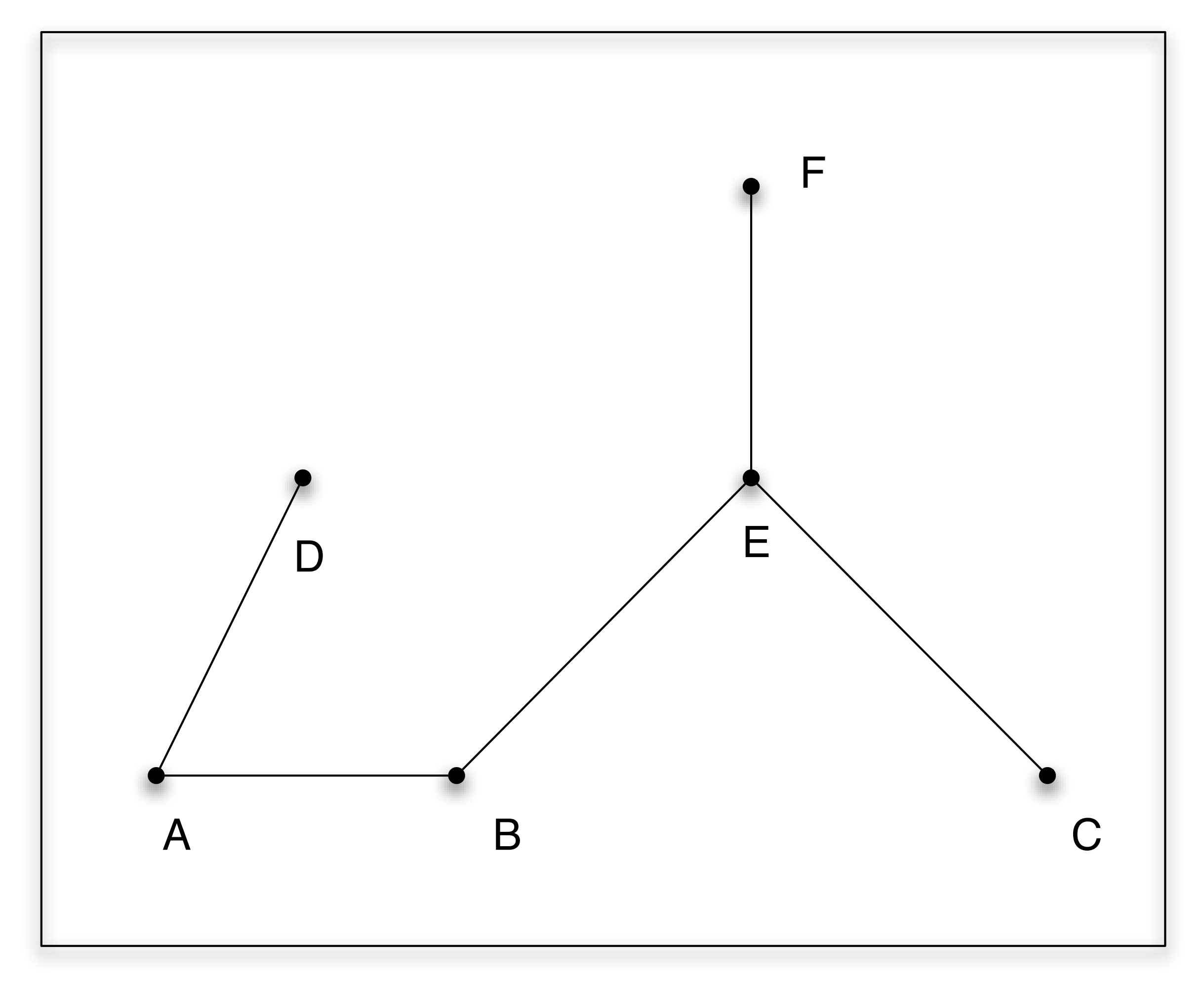


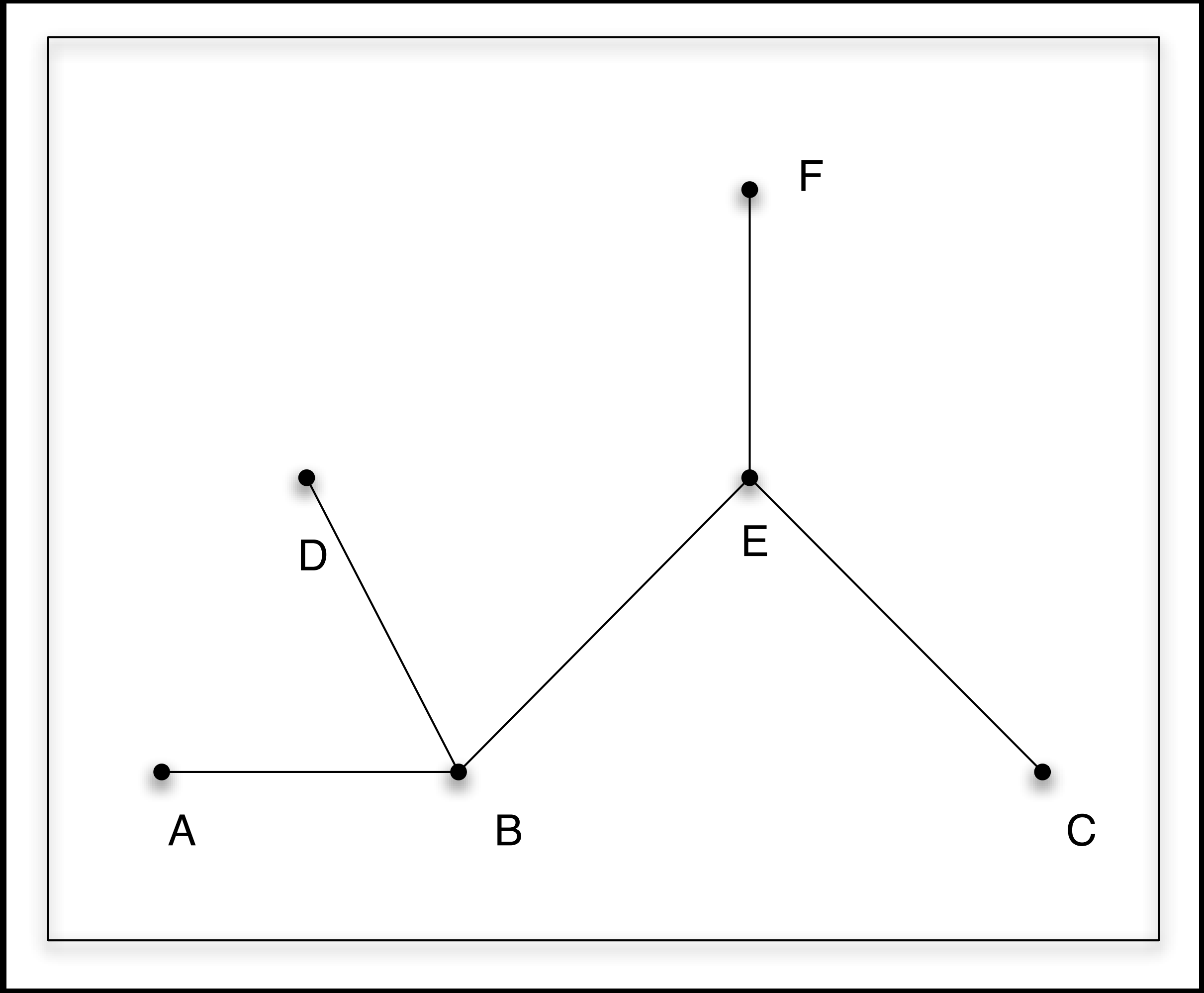
$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Non-directed Graph

Minimum Spanning Tree

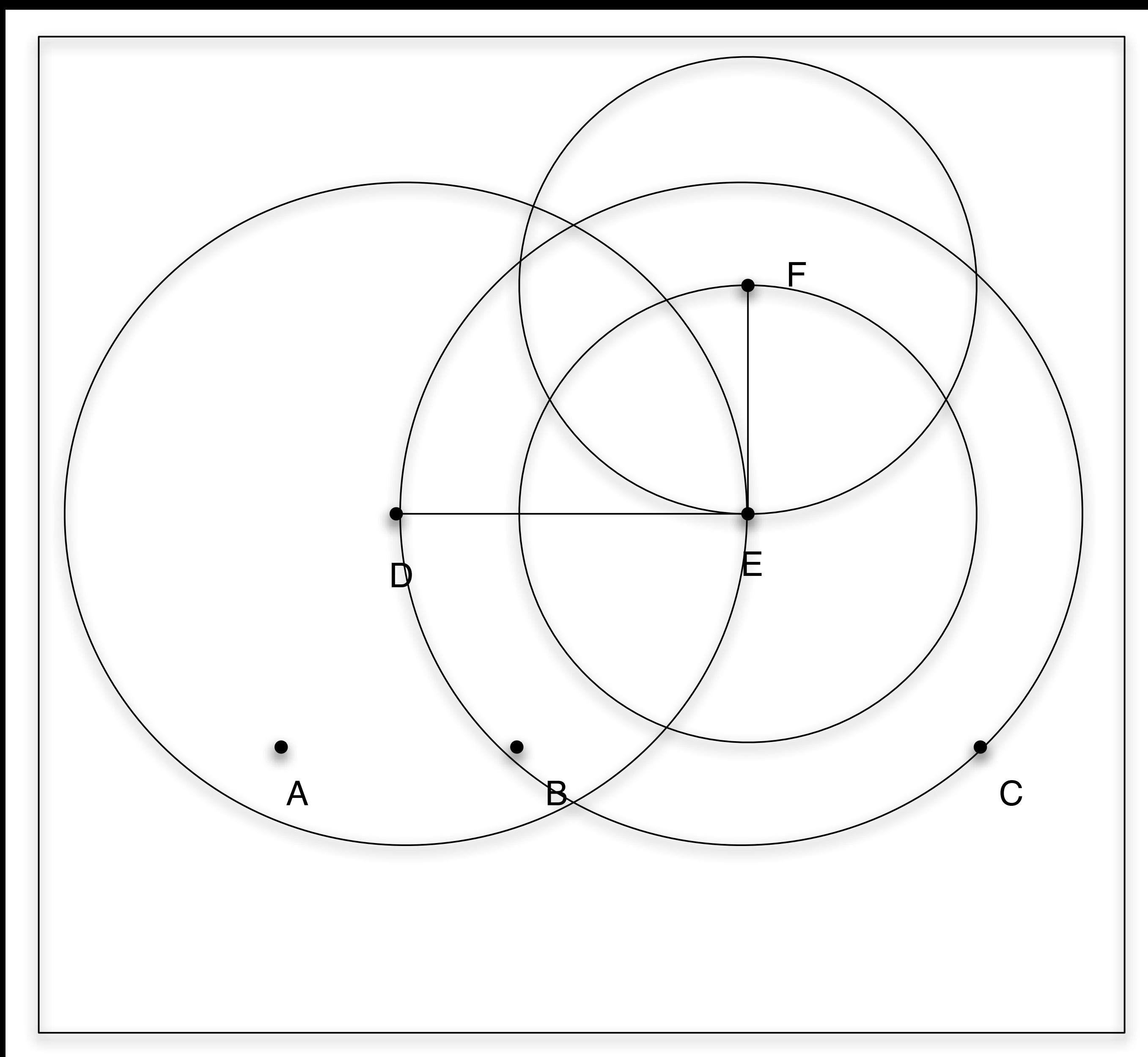
- No Islands
- May not be unique





Relative Neighbors

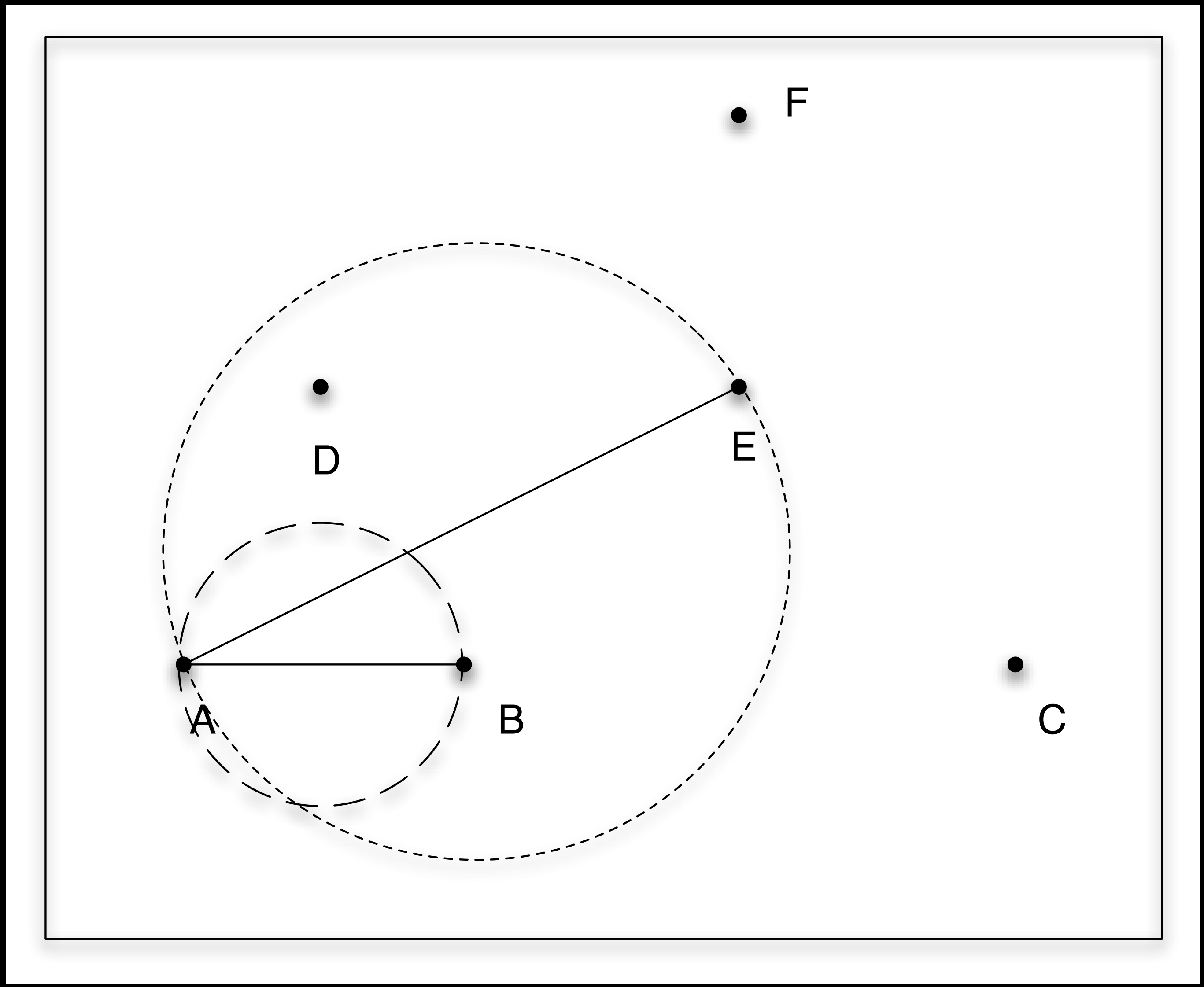
$$d_{i,j} \leq min(max(d_{i,k}, d_{j,k})) \ \forall \ k \neq i, j.$$



$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

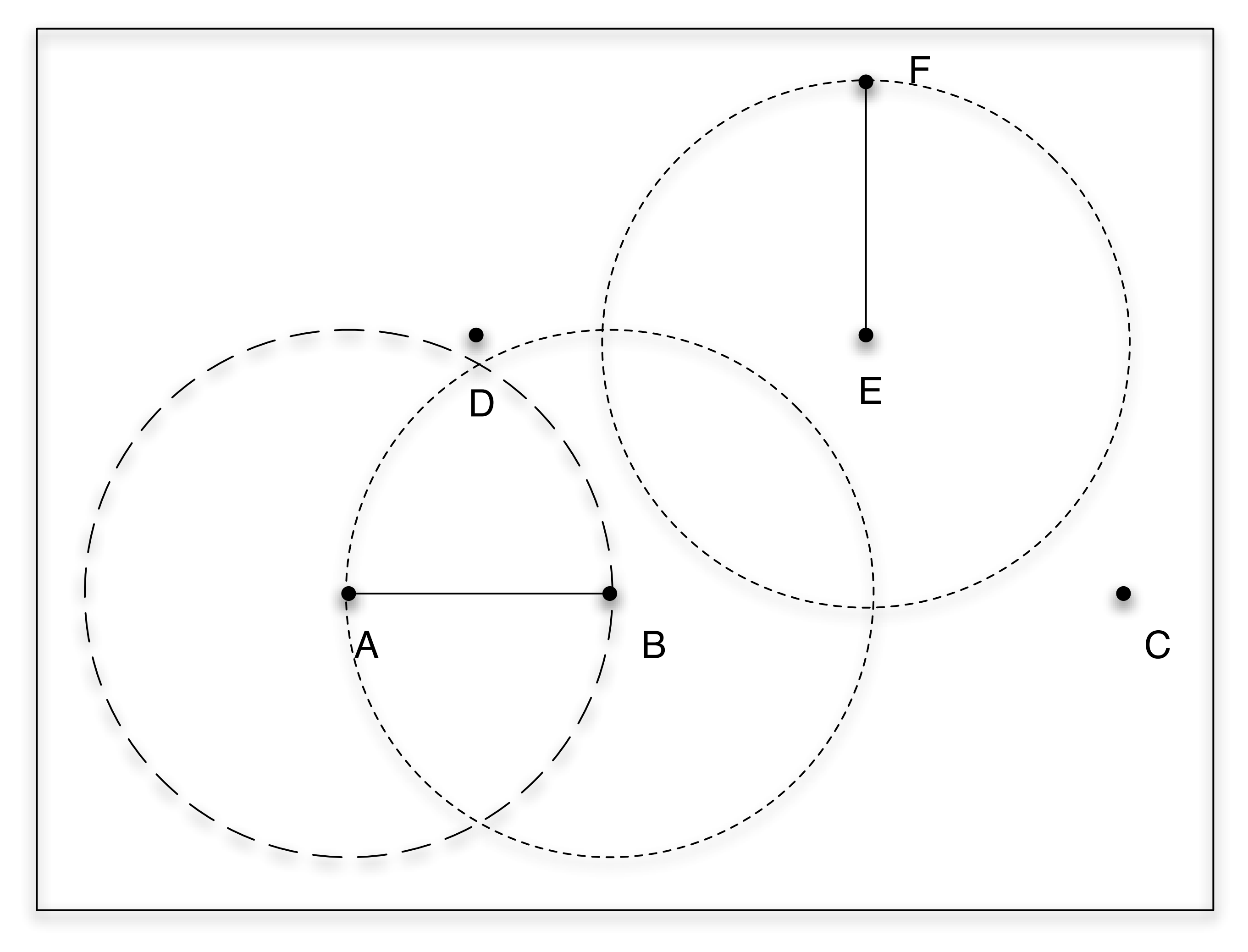
Gabriel Neighbors

$$d_{i,j} \leq min\left(\sqrt{d_{i,k}^2 + d_{j,k}^2}\right)\ \forall\ k \neq i, j.$$



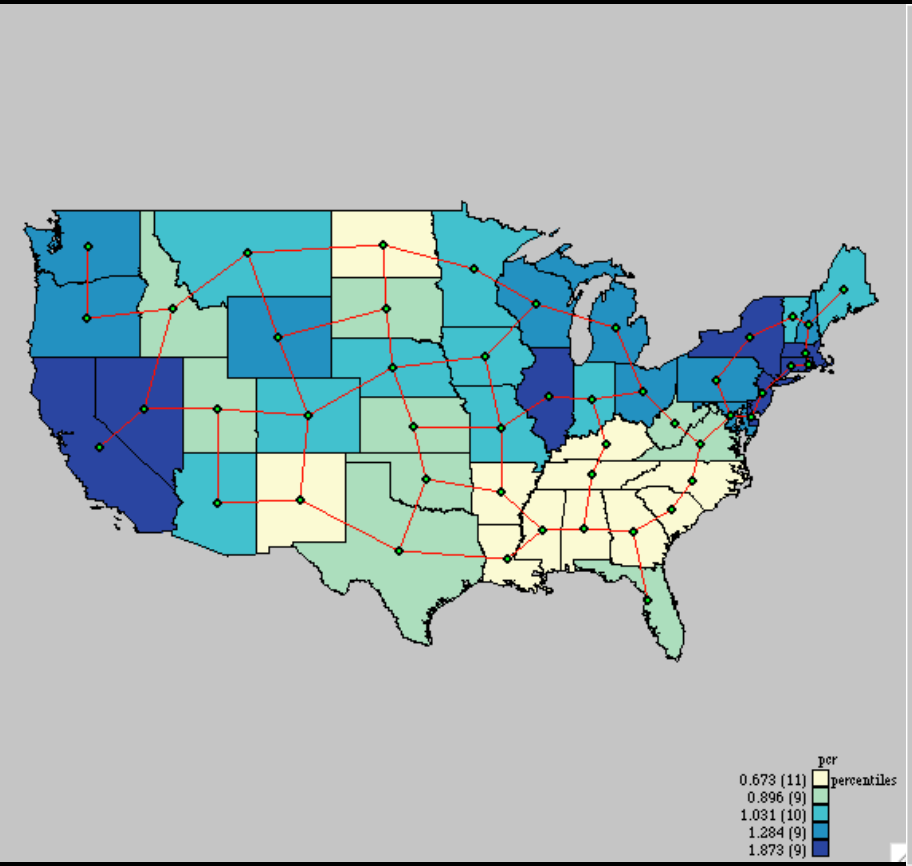
$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Sphere of Influence Neighbors

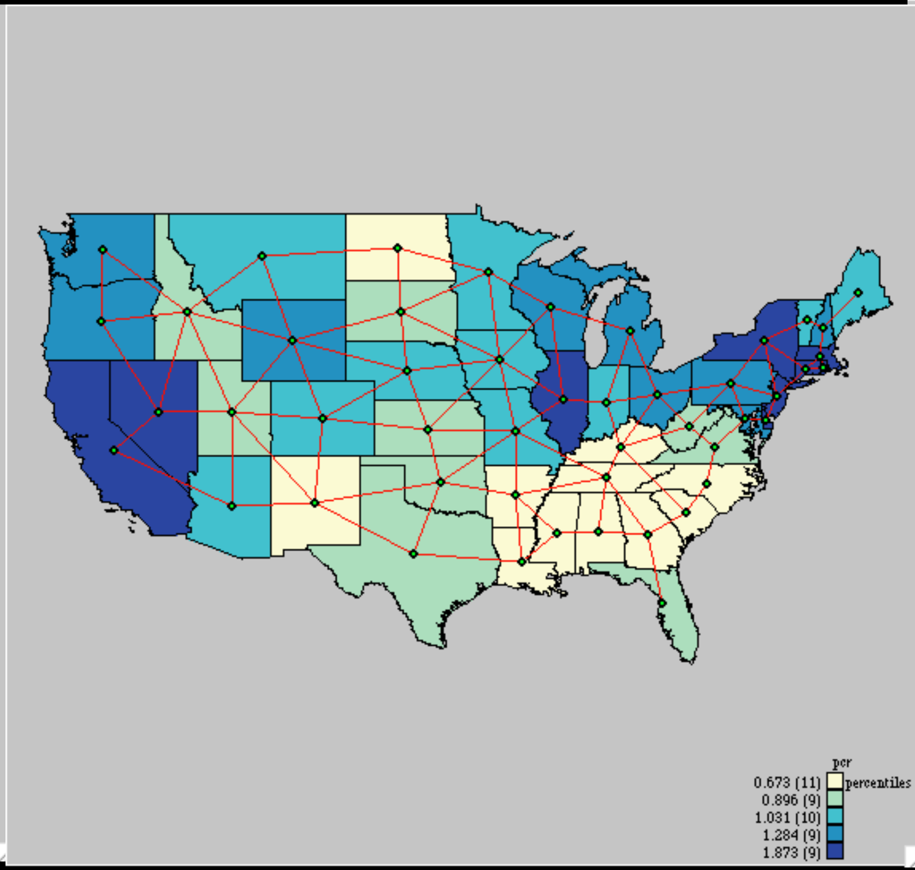


$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

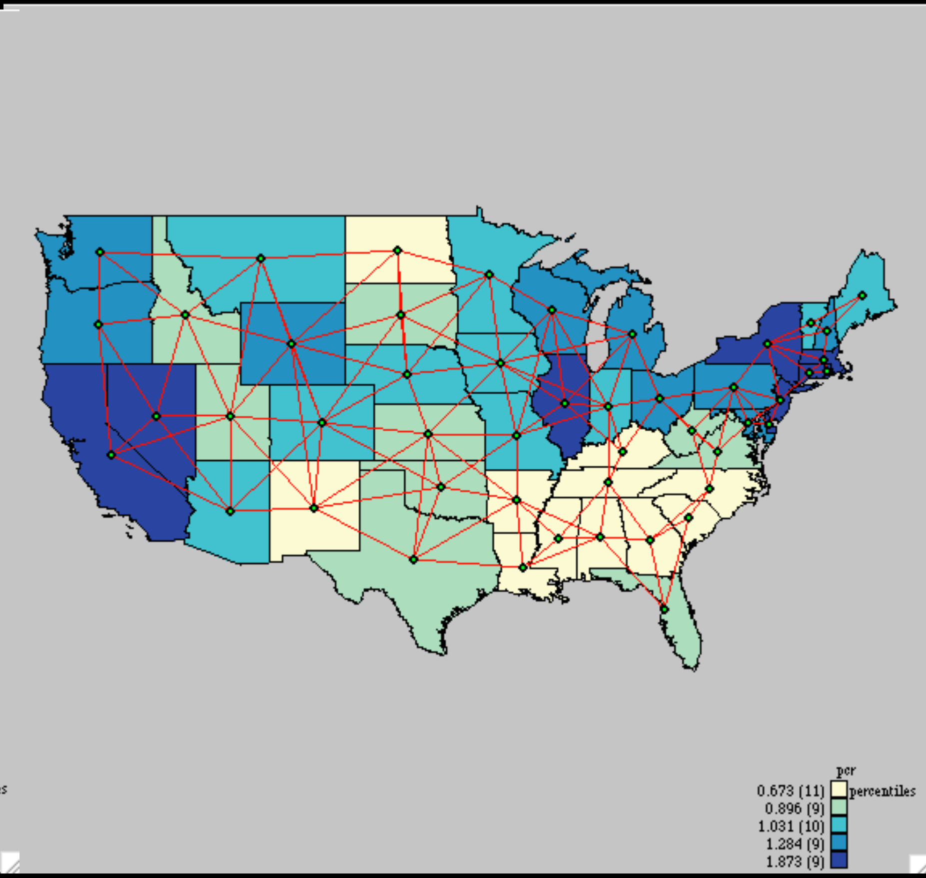
States



Relative

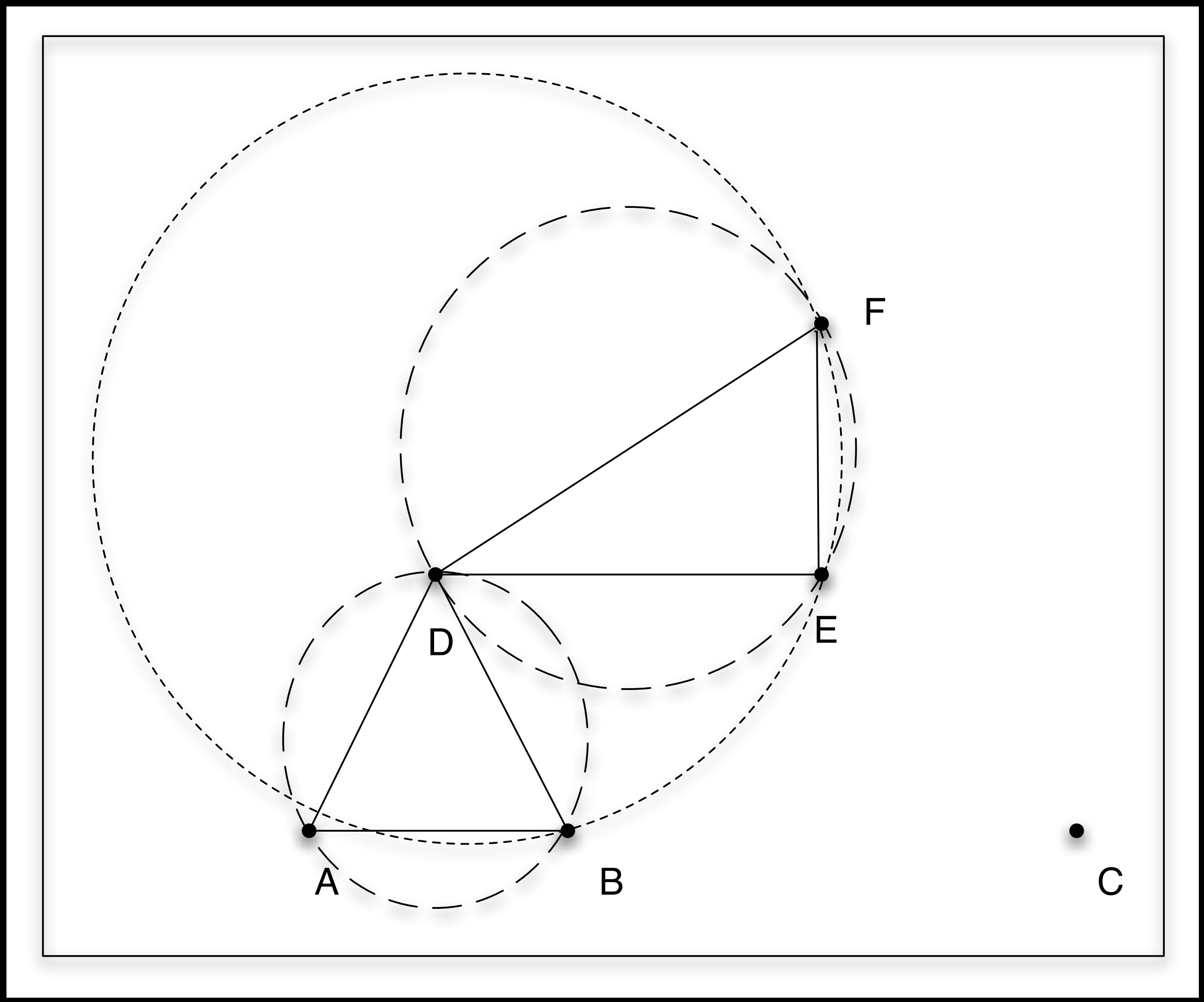


Gabriel



SOI

Delaunay Neighbors



$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Connectivity

Sparse

Dense

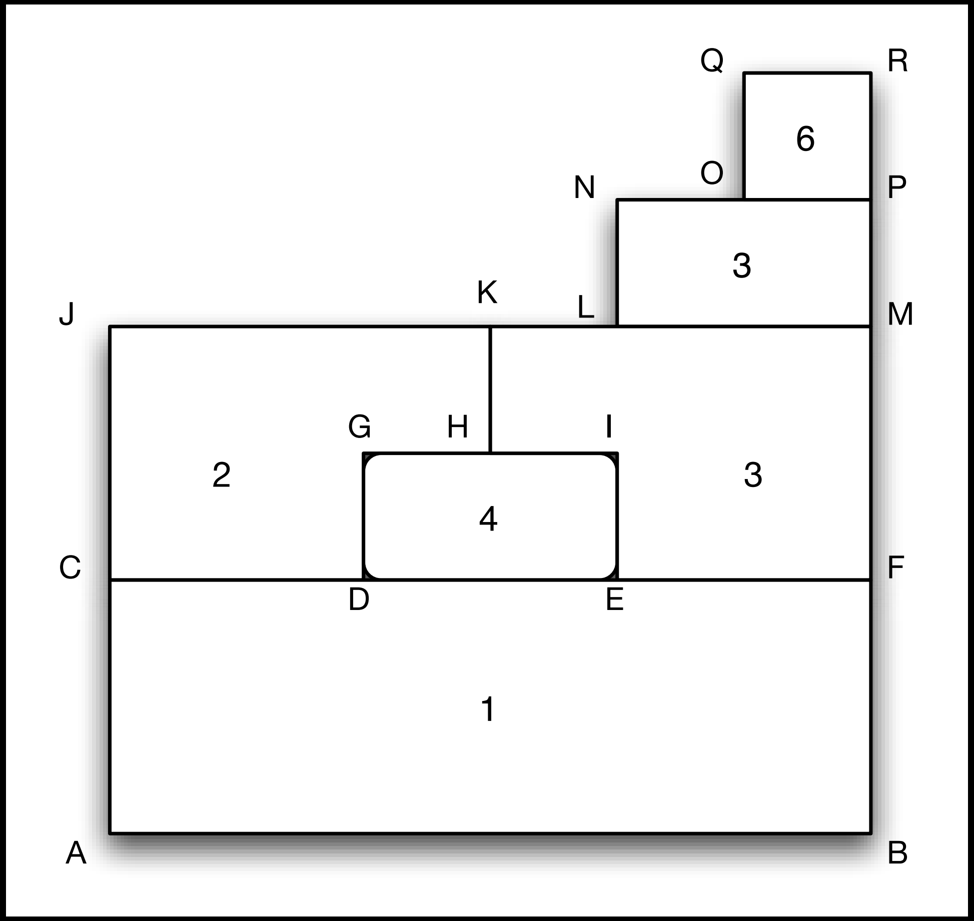
Mutual Nearest Neighbor
Minimum Spanning Tree
Nearest Neighbor
Relative Neighbor
Gabriel
Sphere of Influence
Delaunay

Practical Considerations

- Contiguity not determined via visual inspection
  - Highly error prone
- Specialized software
  - Small problems ($n<1000$) brute force ok
  - Large problems different story

Using GIS to Derive Contiguity

- Topological database
  - ArcInfo Arc Attribute Table

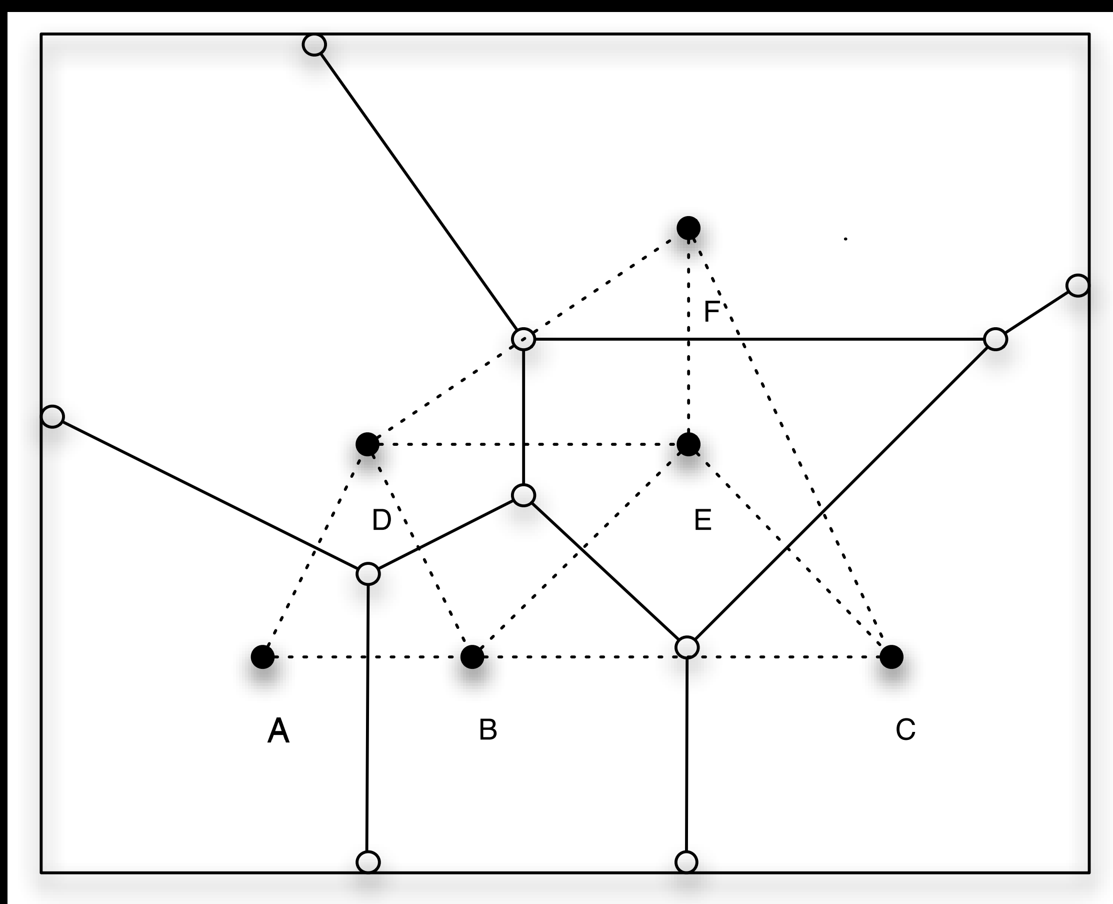


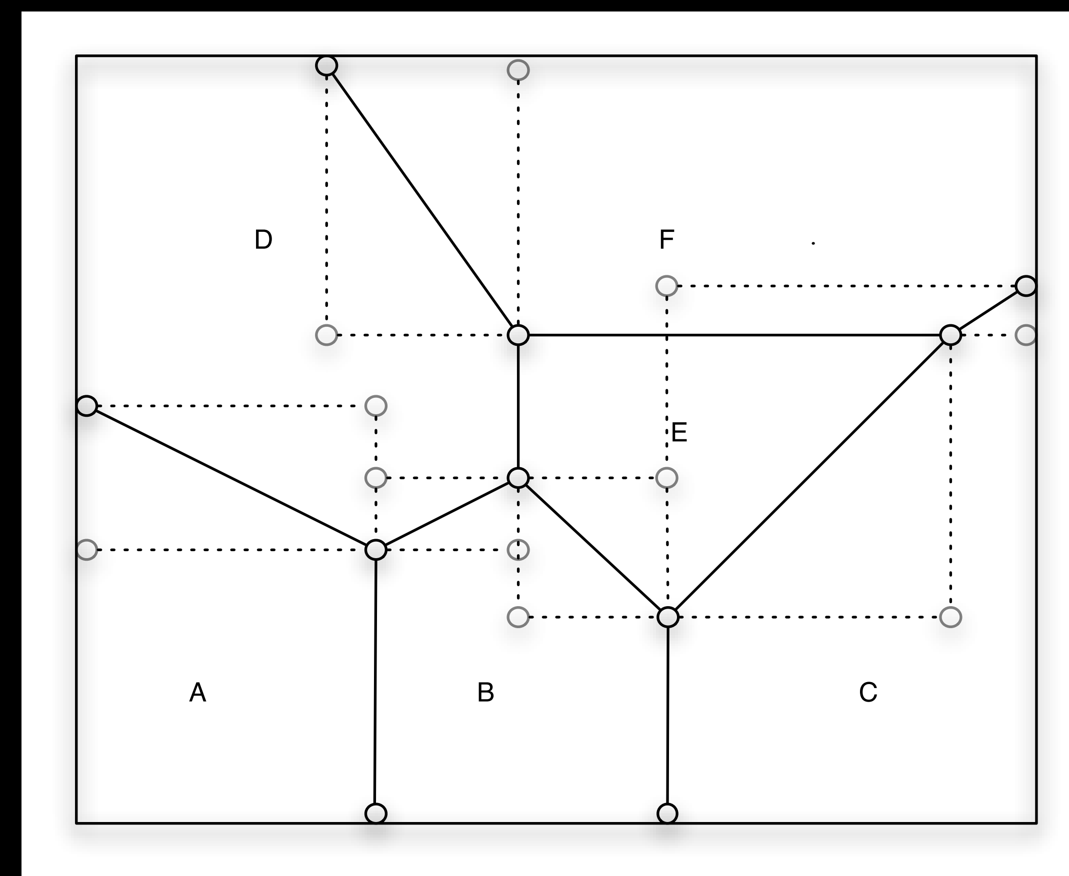
ArcID	FromNode	ToNode	RightPoly	LeftPoly	Length
AB	A	B	0	1	6
BF	B	F	0	1	3
FE	F	E	3	1	2
ED	E	D	4	1	2
DC	D	C	2	1	2
CA	C	A	0	1	3
...	...	...	...	...	...

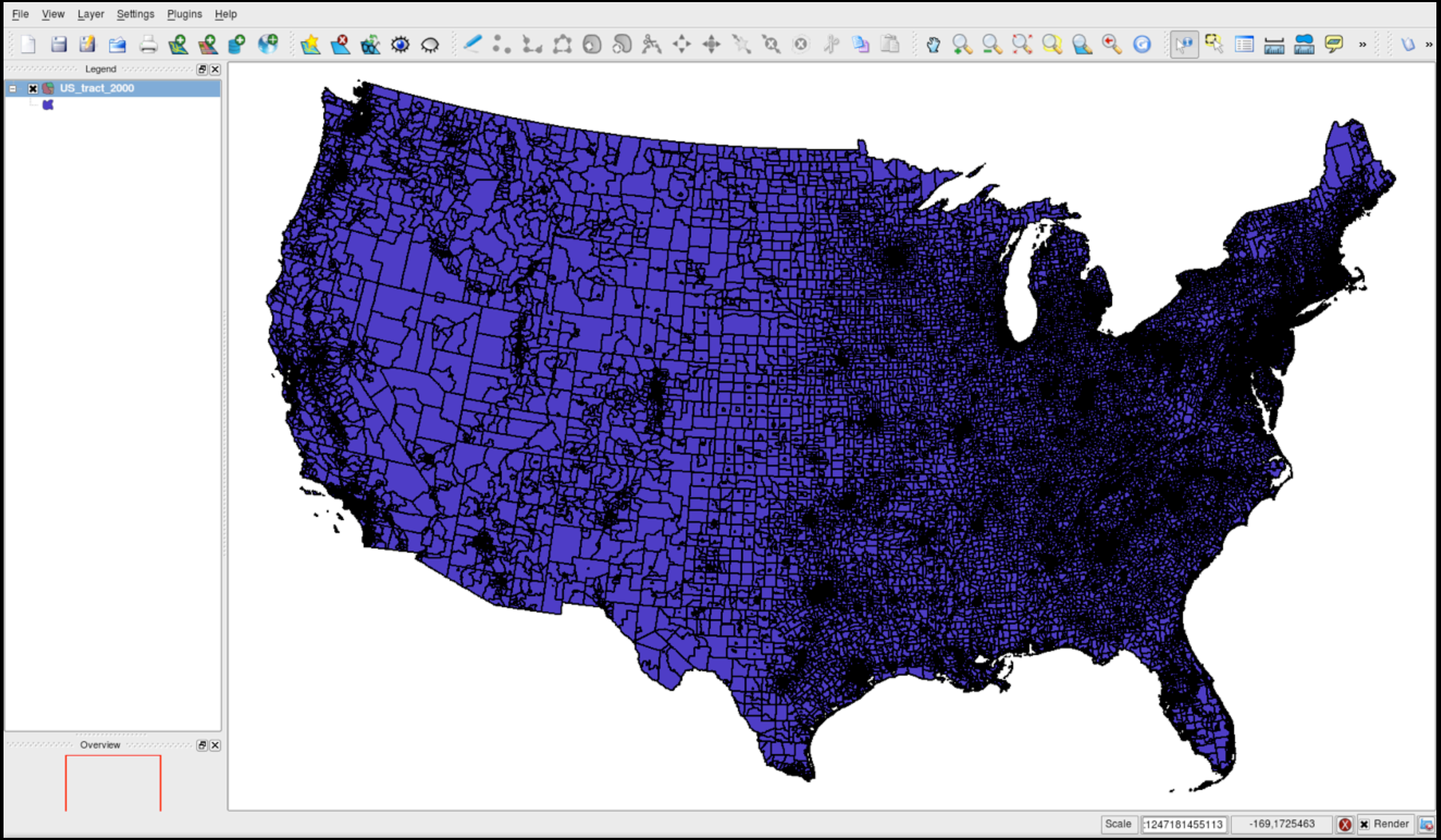
Using GIS to Derive Contiguity

- Nontopological database
  - Shapefile = spaghetti
  - Polygon as a list of vertices
- PySAL
  - Filter-refine + plane sweep algorithm

Minimum Bounding Box to Derive Contiguity






File View Layer Settings Plugins Help
Legend
US_tract_2000
Overview
Scale
1:247181455113
-169,1725463
Render

US Tracts

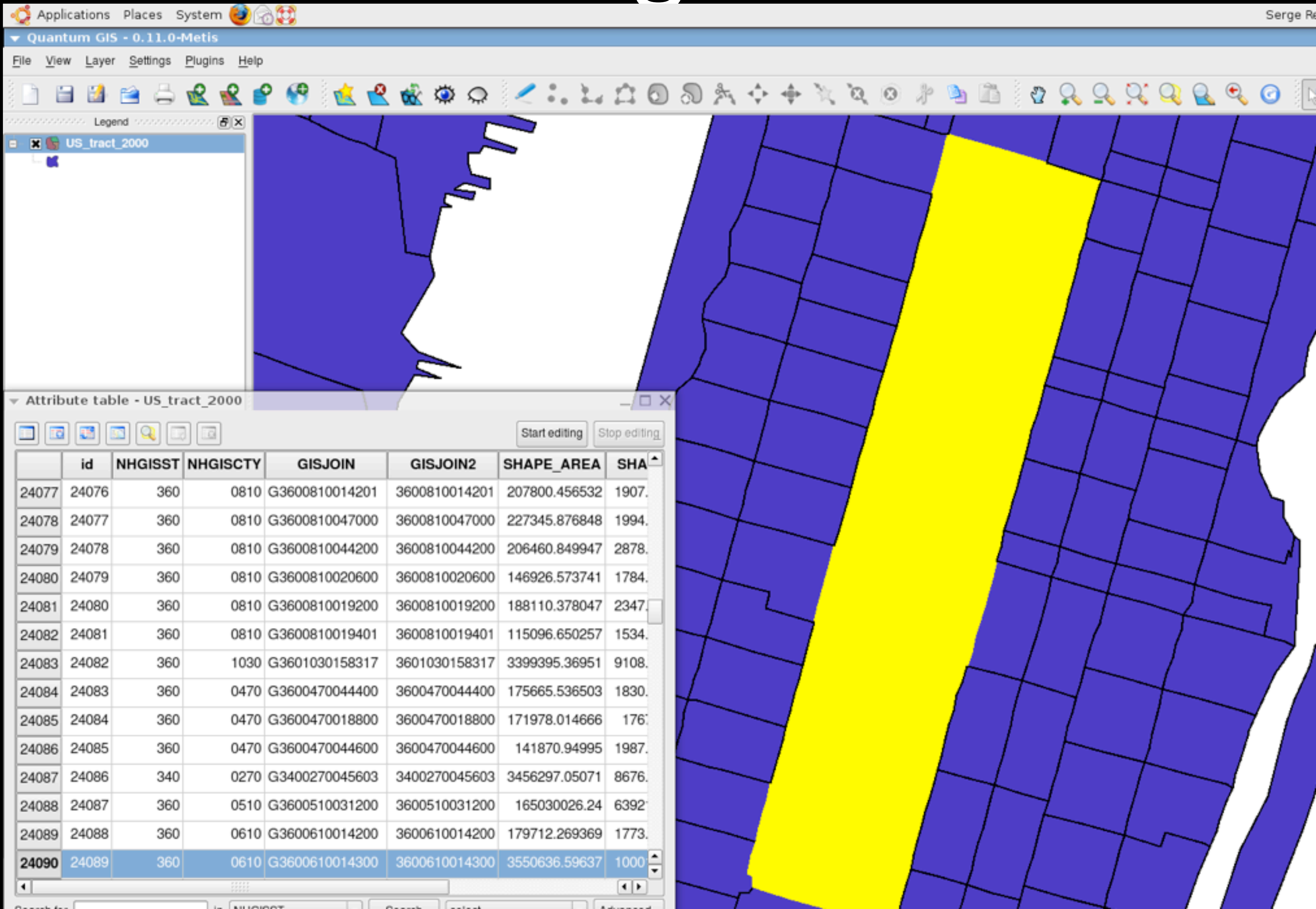
ContiguityWeights

```
fname="/Users/serge/Desktop/nhgis/us_tract_2000/US_tract_2000.shp"
shpFile=shpIO.shpFile(fname)

def run():
    c=ContiguityWeights(shpFile, None, None)

import cProfile
t0=time.time()
cProfile.run('run()')
t1=time.time()
print t1-t0

c=ContiguityWeights(shpFile, None, None)
c.writeGAL("tracts2000.gal")
g=GAL(c.w)
```

Cardinalities

```
>>> cards=[ len(nn) for i,nn in c.w.items()]
>>> max(cards)
30
>>> cards.index(30)
24089
>>> 
```

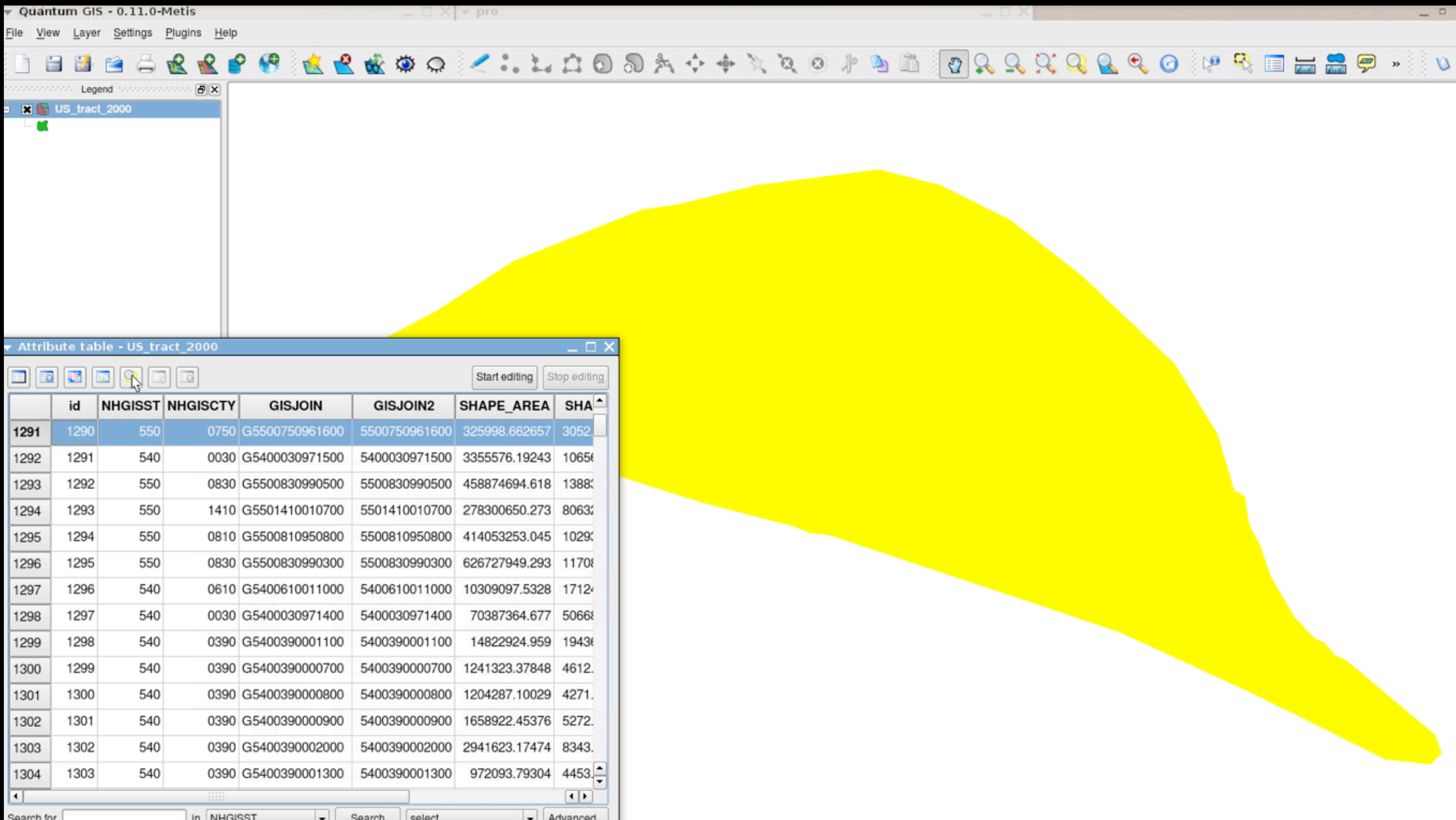
30 neighbors



	id	NHGISST	NHGISCTY	GISJOIN	GISJOIN2	SHAPE_AREA	SHA
24077	24076	360	0810	G3600810014201	3600810014201	207800.456532	1907.
24078	24077	360	0810	G3600810047000	3600810047000	227345.876848	1994.
24079	24078	360	0810	G3600810044200	3600810044200	206460.849947	2878.
24080	24079	360	0810	G3600810020600	3600810020600	146926.573741	1784.
24081	24080	360	0810	G3600810019200	3600810019200	188110.378047	2347.
24082	24081	360	0810	G3600810019401	3600810019401	115096.650257	1534.
24083	24082	360	1030	G3601030158317	3601030158317	3399395.36951	9108.
24084	24083	360	0470	G3600470044400	3600470044400	175665.536503	1830.
24085	24084	360	0470	G3600470018800	3600470018800	171978.014666	176
24086	24085	360	0470	G3600470044600	3600470044600	141870.94995	1987.
24087	24086	340	0270	G3400270045603	3400270045603	3456297.05071	8676.
24088	24087	360	0510	G3600510031200	3600510031200	165030026.24	6392
24089	24088	360	0610	G3600610014200	3600610014200	179712.269369	1773.
24090	24089	360	0610	G3600610014300	3600610014300	3550636.59637	1000

Islands

```
>>> len(g.islands)
46
>>> g.islands[0:10]
[1290, 2859, 2906, 2916, 2996, 3070, 3125, 10554, 10622, 11142]
>>> 
```



	id	NHGISST	NHGISCTY	GISJOIN	GISJOIN2	SHAPE_AREA	SHA
1291	1290	550	0750	G5500750961600	5500750961600	325998.662657	3052.
1292	1291	540	0030	G5400030971500	5400030971500	3355576.19243	1065
1293	1292	550	0830	G5500830990500	5500830990500	458874694.618	1388
1294	1293	550	1410	G5501410010700	5501410010700	278300650.273	8063
1295	1294	550	0810	G5500810950800	5500810950800	414053253.045	1029
1296	1295	550	0830	G5500830990300	5500830990300	626727949.293	1170
1297	1296	540	0610	G5400610011000	5400610011000	10309097.5328	1712
1298	1297	540	0030	G5400030971400	5400030971400	70387364.677	5066
1299	1298	540	0390	G5400390001100	5400390001100	14822924.959	1943
1300	1299	540	0390	G5400390000700	5400390000700	1241323.37848	4612.
1301	1300	540	0390	G5400390000800	5400390000800	1204287.10029	4271.
1302	1301	540	0390	G5400390000900	5400390000900	1658922.45376	5272.
1303	1302	540	0390	G5400390002000	5400390002000	2941623.17474	8343.
1304	1303	540	0390	G5400390001300	5400390001300	972093.79304	4453.

Garbage In

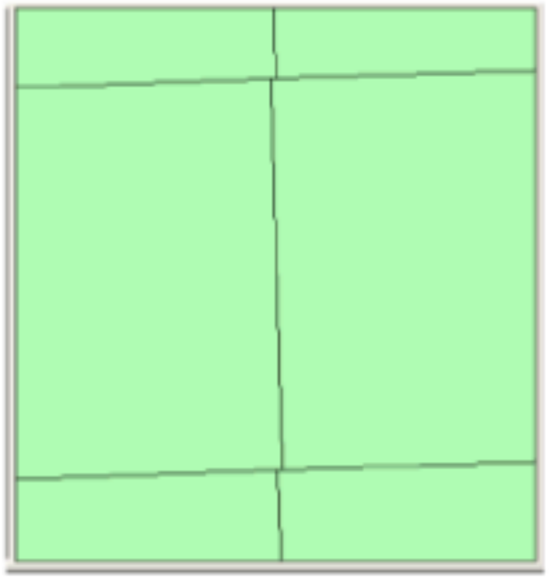


Figure 3.6: Misaligned grid cells

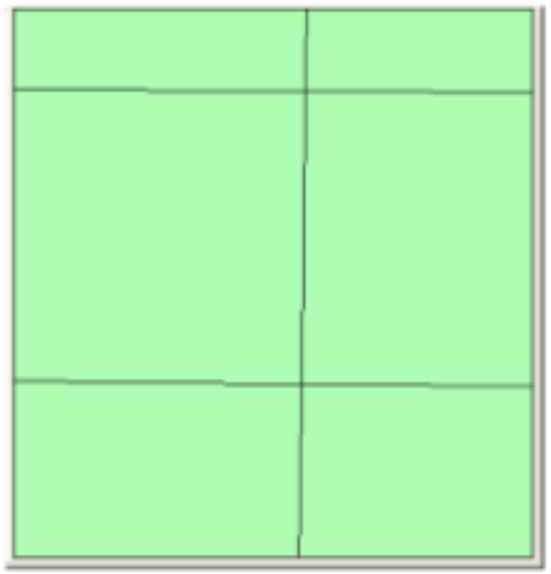


Figure 3.7: Aligned grid cells

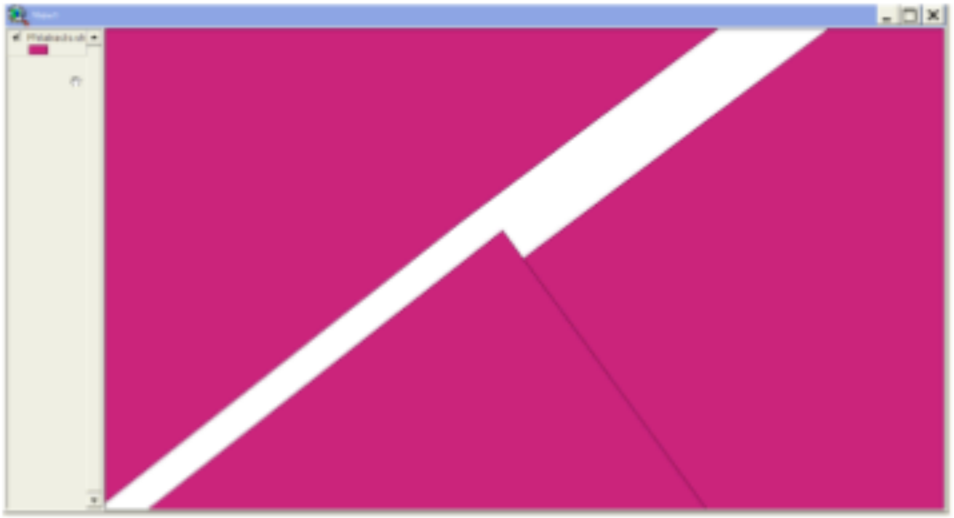


Figure 3.8: Non-touching census tracts

General Weights

Distance Function

- $w_{ij} = f(d_{ij},\Theta)$
- $\partial w_{ij}/\partial d_{ij} < 0$
  - Tobler's first law
  - distance decay effect

Distance Functions

- $w_{ij} = f(d_{ij}, \Theta)$
  - $w_{ij} = 1 / d_{ij}^{\alpha}$
  - $w_{ij} = 0 \ \forall \ d_{ij} > \delta$
  - α
    - α=-1 for inverse distance weights
    - α=-2 for gravity weights

Inverse Distance

$$
\begin{matrix}
0 & 0.100 & 0.033 & 0.089 & 0.045 & 0.035 \\
0.100 & 0 & 0.050 & 0.089 & 0.071 & 0.045 \\
0.033 & 0.050 & 0 & 0.037 & 0.071 & 0.045 \\
0.089 & 0.089 & 0.037 & 0 & 0.067 & 0.056 \\
0.045 & 0.071 & 0.071 & 0.067 & 0 & 0.100 \\
0.035 & 0.045 & 0.045 & 0.056 & 0.100 & 0
\end{matrix}
$$

Gravity Weights

$$
\mathbf{w} = \left| \begin{matrix}
0 & 0.010 & 0.001 & 0.008 & 0.002 & 0.001 \\
0.010 & 0 & 0.003 & 0.008 & 0.005 & 0.002 \\
0.001 & 0.003 & 0 & 0.001 & 0.005 & 0.002 \\
0.008 & 0.008 & 0.001 & 0 & 0.004 & 0.003 \\
0.002 & 0.005 & 0.005 & 0.004 & 0 & 0.010 \\
0.001 & 0.002 & 0.002 & 0.003 & 0.010 & 0
\end{matrix} \right.
$$

Distance-Perimeter Weights

- $w_{ij} = b_{ij}^{\beta} / d_{ij}^{\alpha}$
- b_{ij} is the share of the common border between i and j

Distance-Perimeter

Shared Perimeter

	2	3	4	5	6
1	2	0	2	2	0
2		0	2	1	0
3			0	2	1
4				2	0
5					0

Perimeters

16
10
6
6
10
4

$$\mathbf{b} = \left|\begin{array}{cccccc} 0 & 0.125 & 0 & 0.125 & 0.125 & 0 \\ 0.2 & 0 & 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.333 & 0.167 \\ 0.333 & 0.333 & 0 & 0 & 0.333 & 0 \\ 0.2 & 0.1 & 0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 \end{array}\right.$$

Distance-Perimeter

$$\mathbf{b} = \begin{vmatrix} 0 & 0.125 & 0 & 0.125 & 0.125 & 0 \\ 0.2 & 0 & 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.333 & 0.167 \\ 0.333 & 0.333 & 0 & 0 & 0.333 & 0 \\ 0.2 & 0.1 & 0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 \end{vmatrix}$$

$$\mathbf{W} = \begin{vmatrix} 0 & 0.05 & 0 & 0.08 & 0.05 & 0 \\ 0.08 & 0 & 0 & 0.13 & 0.07 & 0 \\ 0 & 0 & 0 & 0 & 0.21 & 0.15 \\ 0.22 & 0.21 & 0 & 0 & 0.21 & 0 \\ 0.08 & 0.03 & 0.13 & 0.13 & 0 & 0 \\ 0 & 0 & 0.23 & 0 & 0 & 0 \end{vmatrix}$$

Kernel Weights

- $w_{ij} = K(u/h)$
  - K: kernel, h: bandwidth
  - $k(u/h) = 0$ for $u > h$
  - K
    - $\int K(u)\,du = 1$

Kernel Functions

Functions	Expression
Truncated, Uniform	$K(z) = 1 \times \mathbf{1}(\lvert z\rvert \leq 1)$
Triangular, Bartlett	$K(z) = (1 - \lvert z\rvert) \times \mathbf{1}(\lvert z\rvert \leq 1)$
Epanechnikov, Quadratic	$K(z) = (3/4)(1 - z^2) \times \mathbf{1}(\lvert z\rvert \leq 1)$
Quartic, Biweight, Bisquare	$K(z) = (15/16)(1 - z^2)^2 \times \mathbf{1}(\lvert z\rvert \leq 1)$
Parzen	$K(z) = (1 - 6z^2 + 6\lvert z\rvert^3) \times \mathbf{1}(0 \leq \lvert z\rvert \leq 1/2)$
	$K(z) = 2(1 - \lvert z\rvert)^3) \times \mathbf{1}(1/2 < \lvert z\rvert \leq 1)$
Gaussian, Normal	$K(z) = (2\pi)^{-1/2} \exp(-z^2/2)$

Kernel δ = 15.0

Arguments

	B	C	D	E	F
A	0.667	—	0.747	—	—
B		—	0.747	0.940	—
C			—	0.940	—
D				1.000	—
E					0.667

$$\mathbf{W} = \left|\begin{array}{cccccc} 1.000 & 0.333 & 0 & 0.253 & 0 & 0 \\ 0.333 & 1.000 & 0 & 0.253 & 0.060 & 0 \\ 0 & 0 & 1.000 & 0 & 0.060 & 0 \\ 0.253 & 0.253 & 0 & 1.000 & 0.000 & 0 \\ 0 & 0.060 & 0.060 & 0.000 & 1.000 & 0.333 \\ 0 & 0 & 0 & 0 & 0.333 & 1.000 \end{array}\right.$$

Statistical Weights

- AMOEBA (Aldstatd and Getis 2006)

$$w_{ij} = \frac{P[G_i^*(k_{max})] - P[G_i^*(k_j)]}{P[G_i^*(k_{max})] - P[G_i^*(0)]}$$

where $P[\,]$ is the cumulative probability of a standard normal variate.

Socioeconomic Weights

- Many Different Types
  - Socioeconomic Similarity
  - Input-Output Tables
  - Social Networks
- Issues
  - W now endogenous
  - Regularity conditions may be violated

Row Standardization

$$w^*_{ij} = \frac{w_{ij}}{\sum_j w_{ij}}$$

Number of Neighbors

$$\mathbf{K} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Standardization

$$W^* = K^{-1}W$$

$$\mathbf{W}^* = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{W}^* = \begin{bmatrix} 0 & 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Guidance

How to Select the Right W?

- Theoretical priors
  - exceptionally rare
- Typically no "best" solution
- Relation to spatial interaction theory

Rules of Thumb (Griffith 1996)

- “It is better to posit some reasonable geographic weights matrix specification than to assume all entries are zero.
- “..use a surface partitioning that falls somewhere between a regular square and a regular hexagonal tesselation”
- “ ..better to employ a somewhat under-specified than a somewhat over-specified geographic weights matrix.”

Choice of W

- For testing for autocorrelation
  - power matters
  - contiguity may be sufficient
  - power robust to misspecification
- For modeling and parameter estimation
  - more complex and fragile